1 TRIGONOMETRY

Objectives

After studying this chapter you should

- be able to handle with confidence a wide range of trigonometric identities;
- be able to express linear combinations of sine and cosine in any of the forms $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$;
- know how to find general solutions of trigonometric equations;
- be familiar with inverse trigonometric functions and the associated calculus.

1.0 Introduction

In the first *Pure Mathematics* book in this series, you will have encountered many of the elementary results concerning the trigonometric functions. These will, by and large, be taken as read in this chapter. However, in the first few sections there is some degree of overlap between the two books: this will be good revision for you.

1.1 Sum and product formulae

You may recall that

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

Adding these two equations gives

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B \qquad (1)$$

Let $C = A+B$ and $D = A-B$,

then $C+D = 2A$ and $C-D = 2B$. Hence

$$A = \frac{C+D}{2},\ B = \frac{C-D}{2}$$

and (1) can be written as

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

This is more easily remembered as

'sine plus sine = twice sine(half the sum) cos(half the difference)'

Activity 1

In a similar way to above, derive the formulae for

(a) $\sin C - \sin D$ (b) $\cos C + \cos D$ (c) $\cos C - \cos D$

By reversing these formulae, write down further formulae for

(a) $2\sin E\cos F$ (b) $2\cos E\cos F$ (c) $2\sin E\sin F$

Example

Show that $\cos 59° + \sin 59° = \sqrt{2}\cos 14°$.

Solution

Firstly, $\sin 59° = \cos 31°$, since $\sin\theta = \cos(90-\theta)$

So

$$\begin{aligned}
\text{LHS} &= \cos 59° + \cos 31° \\
&= 2\cos\left(\frac{59+31}{2}\right)\cos\left(\frac{59-31}{2}\right) \\
&= 2\cos 45° \times \cos 14° \\
&= 2 \times \frac{\sqrt{2}}{2}\cos 14° \\
&= \sqrt{2}\cos 14° \\
&= \text{RHS}
\end{aligned}$$

Example

Prove that $\sin x + \sin 2x + \sin 3x = \sin 2x(1 + 2\cos x)$.

Solution

$$\begin{aligned}
\text{LHS} &= \sin 2x + (\sin x + \sin 3x) \\
&= \sin 2x + 2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) \\
&= \sin 2x + 2\sin 2x\cos x \\
&= \sin 2x(1 + 2\cos x)
\end{aligned}$$

Example

Write $\cos 4x \cos x - \sin 6x \sin 3x$ as a product of terms.

Solution

Now $\cos 4x \cos x = \frac{1}{2}\{\cos(4x+x)+\cos(4x-x)\}$

$$= \frac{1}{2}\cos 5x + \frac{1}{2}\cos 3x$$

and $\sin 6x \sin 3x = \frac{1}{2}\{\cos(6x-3x)-\cos(6x+3x)\}$

$$= \frac{1}{2}\cos 3x - \frac{1}{2}\cos 9x$$

Thus, $\text{LHS} = \frac{1}{2}\cos 5x + \frac{1}{2}\cos 3x - \frac{1}{2}\cos 3x + \frac{1}{2}\cos 9x$

$$= \frac{1}{2}(\cos 5x + \cos 9x)$$

$$= \frac{1}{2} \times 2\cos\left(\frac{5x+9x}{2}\right)\cos\left(\frac{5x-9x}{2}\right)$$

$$= \cos 7x \cos 2x$$

The sum formulae are given by

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

and the product formulae by

$$\sin A\cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))$$
$$\cos A\cos B = \frac{1}{2}(\cos(A+B)+\cos(A-B))$$
$$\sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B))$$

Exercise 1A

1. Write the following expressions as products:

   (a) $\cos 5x - \cos 3x$ (b) $\sin 11x - \sin 7x$

   (c) $\cos 2x + \cos 9x$ (d) $\sin 3x + \sin 13x$

   (e) $\cos\frac{2\pi}{15} + \cos\frac{14\pi}{15} + \cos\frac{4\pi}{15} + \cos\frac{8\pi}{15}$

   (f) $\sin 40° + \sin 50° + \sin 60°$

   (g) $\cos 114° + \sin 24°$

2. Evaluate in rational/surd form

   $\sin 75° + \sin 15°$

3. Write the following expressions as sums or differences:

   (a) $2\cos 7x \cos 5x$

   (b) $2\cos\left(\frac{1}{2}x\right)\cos\left(\frac{5x}{2}\right)$

   (c) $2\sin\left(\frac{\pi}{4} - 3\theta\right)\cos\left(\frac{\pi}{4} + \theta\right)$

   (d) $2\sin 165° \cos 105°$

4. Establish the following identities:

   (a) $\cos\theta - \cos 3\theta = 4\sin^2\theta\cos\theta$

   (b) $\sin 6x + \sin 4x - \sin 2x = 4\cos 3x \sin 2x \cos x$

   (c) $\dfrac{2\sin 4A + \sin 6A + \sin 2A}{2\sin 4A - \sin 6A - \sin 2A} = \cot^2 A$

   (d) $\dfrac{\sin(A+B) + \sin(A-B)}{\cos(A+B) + \cos(A-B)} = \tan A$

   (e) $\dfrac{\cos(\theta + 30°) + \cos(\theta + 60°)}{\sin(\theta + 30°) + \sin(\theta + 60°)} = \dfrac{1 - \tan\theta}{1 + \tan\theta}$

5. Write $\cos 12x + \cos 6x + \cos 4x + \cos 2x$ as a product of terms.

6. Express $\cos 3x \cos x - \cos 7x \cos 5x$ as a product of terms.

1.2 Linear combinations of sin and cos

Expressions of the form $a\cos\theta + b\sin\theta$, for constants a and b, involve two trig functions which, on the surface, makes them difficult to handle. After working through the following activity, however, you should be able to see that such expressions (called **linear combinations** of sin and cos – linear since they involve no squared terms or higher powers) can be written as a single trig function. By re-writing them in this way you can deduce many results from the elementary properties of the sine or cosine function, and solve equations, without having to resort to more complicated techniques.

For this next activity you will find it very useful to have a graph plotting facility. Remember, you will be working in radians.

Activity 2

Sketch the graph of a function of the form

$$y = a\sin x + b\cos x$$

(where a and b are constants) in the range $-\pi \le x \le \pi$.

From the graph, you must identify the amplitude of the function and the x-coordinates of

(i) the crossing point on the x-axis nearest to the origin, and

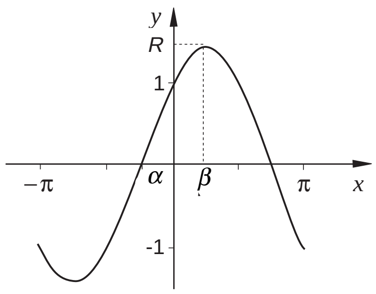
(ii) the first maximum of the function

as accurately as you can.

An example has been done for you; for $y = \sin x + \cos x$, you can see that amplitude $R \approx 1.4$

crossing point nearest to the origin O at $x = \alpha = -\frac{\pi}{4}$

maximum occurs at $x = \beta = \frac{\pi}{4}$



Try these for yourself :

(a) $y = 3\sin x + 4\cos x$ (b) $y = 12\cos x - 5\sin x$

(c) $y = 9\cos x + 12\sin x$ (d) $y = 15\sin x - 8\cos x$

(e) $y = 2\sin x + 5\cos x$ (f) $y = 3\cos x - 2\sin x$

In each case, make a note of

R, the amplitude;
α, the crossing point nearest to O;
β, the x-coordinate of the maximum.

In each example above, you should have noticed that the curve is itself a sine/cosine 'wave'. These can be obtained from the curves of either $y = \sin x$ or $y = \cos x$ by means of two simple transformations (taken in any order).

1. A **stretch** parallel to the y-axis by a factor of R, the amplitude, and

2. A **translation** parallel to the x-axis by either α or β (depending on whether you wish to start with sin x or cos x as the original function).

Consider, for example $y = \sin x + \cos x$. This can be written in the form $y = R\sin(x + \alpha)$, since

$$R\sin(x+\alpha) = R\{\sin x \cos\alpha + \cos x \sin\alpha\}$$
$$= R\cos\alpha \sin x + R\sin\alpha \cos x$$

The $R(>0)$ and α should be chosen so that this expression is the same as $\sin x + \cos x$.

Thus

$$R\cos\alpha = 1 \text{ and } R\sin\alpha = 1$$

Dividing these terms gives

$$\tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Squaring and adding the two terms gives

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 1^2$$

$$R^2\left(\cos^2\alpha + \sin^2\alpha\right) = 2$$

Since $\cos^2\alpha + \sin^2\alpha = 1$,

$$R^2 = 2 \Rightarrow R = \sqrt{2}$$

Thus

$$\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$

Activity 3

Express the function $\sin x + \cos x$ in the form

$$\sin x + \cos x = R\cos(x - \alpha)$$

Find suitable values for R and α using the method shown above.

Another way of obtaining the result in Activity 3 is to note that

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

so that

$$\begin{aligned}\sin x + \cos x &= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\\ &= \sqrt{2}\cos\left(\frac{\pi}{2} - \left(x + \frac{\pi}{4}\right)\right)\\ &= \sqrt{2}\cos\left(\frac{\pi}{4} - x\right)\\ &= \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)\end{aligned}$$

since $\cos(-\theta) = \cos\theta$.

Example

Write $7\sin x - 4\cos x$ in the form $R\sin(x-\alpha)$

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Solution

Assuming the form of the result,

$$\begin{aligned} 7\sin x - 4\cos x &= R\sin(x-\alpha) \\ &= R\sin x\cos\alpha - R\cos x\sin\alpha \end{aligned}$$

To satisfy the equation, you need

$$R\cos\alpha = 7$$

$$R\sin\alpha = 4$$

Squaring and adding, as before, gives

$$R = \sqrt{7^2 + 4^2} = \sqrt{65}$$

Thus

$$\cos\alpha = \frac{7}{\sqrt{65}}, \quad \sin\alpha = \frac{4}{\sqrt{65}} \quad \left(\text{or } \tan\alpha = \frac{4}{7}\right)$$

$$\Rightarrow \alpha = 0.519 \text{ radians, to 3 sig. figs.}$$

so $$7\sin x - 4\cos x = \sqrt{65}\sin(x - 0.519)$$

Exercise 1B

Write (in each case, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$)

1. $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$
2. $4\cos x + 3\sin x$ in the form $R\cos(x-\alpha)$
3. $15\sin x - 8\cos x$ in the form $R\sin(x-\alpha)$
4. $6\cos x - 2\sin x$ in the form $R\cos(x+\alpha)$
5. $20\sin x - 21\cos x$ in the form $R\sin(x-\alpha)$
6. $14\cos x + \sin x$ in the form $R\cos(x-\alpha)$
7. $2\cos 2x - \sin 2x$ in the form $R\cos(2x+\alpha)$
8. $3\cos\frac{1}{2}x + 5\sin\frac{1}{2}x$ in the form $R\sin\left(\frac{1}{2}x+\alpha\right)$

1.3 Linear trigonometric equations

In this section you will be looking at equations of the form

$$a\cos x + b\sin x = c$$

for given constants *a*, *b* and *c*.

Example

Solve $3\cos x + \sin x = 2$ for $0° \le x \le 360°$.

Solution

Method 1

Note that $\cos^2 x$ and $\sin^2 x$ are very simply linked using $\cos^2 x + \sin^2 x = 1$ so a 'rearranging and squaring' approach would seem in order.

Rearranging: $$3\cos x = 2 - \sin x$$

Squaring: $$9\cos^2 x = 4 - 4\sin x + \sin^2 x$$

$$\Rightarrow \quad 9\left(1 - \sin^2 x\right) = 4 - 4\sin x + \sin^2 x$$

$$\Rightarrow \quad 0 = 10\sin^2 x - 4\sin x - 5$$

The quadratic formula now gives $\sin x = \dfrac{4 \pm \sqrt{216}}{20}$

and $\quad \sin x \approx 0.93485$ or -0.534847

giving $\quad x = 69.2°, 110.8°$ or $212.3°, 327.7°$ (1 d.p.)

Method 2

Write $3\cos x + \sin x$ as $R\cos(x - \alpha)$ (or $R\sin(x - \alpha)$)

$$3\cos x + \sin x = R\cos(x - \alpha)$$

Firstly, $\quad R = \sqrt{3^2 + 1^2} = \sqrt{10}$

so $\quad 3\cos x + \sin x = \sqrt{10}\left(\dfrac{3}{\sqrt{10}}\cos x + \dfrac{1}{\sqrt{10}}\sin x\right)$

$$\equiv \sqrt{10}\left(\cos x \cos\alpha + \sin x \sin\alpha\right)$$

Thus $\cos\alpha = \dfrac{3}{\sqrt{10}} \left(\text{or } \sin\alpha = \dfrac{7}{\sqrt{10}} \text{ or } \tan\alpha = \dfrac{1}{3} \right) => \alpha = 18.43°$

The equation $3\cos x + \sin x = 2$ can now be written as

$$\sqrt{10}\cos(x - 18.43°) = 2$$

$$\Rightarrow \qquad \cos(x - 18.43°) = \frac{2}{\sqrt{10}}$$

$$\Rightarrow \qquad x - 18.43° = \cos^{-1}\left(\frac{2}{\sqrt{10}}\right)$$

$$\Rightarrow \qquad x - 18.43° = 50.77° \text{ or } 309.23°$$

and

$$x = 50.77° + 18.43° \text{ or } 309.23° + 18.43°$$

$$x = 69.2° \text{ or } 327.7° \quad \text{(1 d.p.)}$$

The question now arises as to why one method yields four answers, the other only two. If you check all four answers you will find that the two additional solutions in Method 1 do not fit the equation $3\cos x + \sin x = 2$. They have arisen as extra solutions created by the squaring process. (Think of the difference between the equations $x = 2$ and $x^2 = 4$: the second one has two solutions.) If Method 1 is used, then the final answers always need to be checked in order to discard the extraneous solutions.

Exercise 1C

1. By writing $7\sin x + 6\cos x$ in the form $R\sin(x + \alpha)(R > 0, 0° < \alpha < 90°)$ solve the equation $7\sin x + 6\cos x = 9$ for values of x between $0°$ and $360°$.

2. Use the 'rearranging and squaring' method to solve

   (a) $4\cos\theta + 3\sin\theta = 2$

   (b) $3\sin\theta - 2\cos\theta = 1$

   for $0° \le \theta \le 360°$.

3. Write $\sqrt{3}\cos\theta + \sin\theta$ as $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$ and hence solve $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$ for $0 \le \theta \le 2\pi$.

4. Solve

   (a) $7\cos x - 6\sin x = 4$ for $-180° \le x \le 180°$

   (b) $6\sin\theta + 8\cos\theta = 7$ for $0° \le \theta \le 180°$

   (c) $4\cos x + 2\sin x = \sqrt{5}$ for $0° \le x \le 360°$

   (d) $\sec x + 5\tan x + 12 = 0$ for $0 \le x \le 2\pi$

1.4 More demanding equations

In this section you will need to keep in mind all of the identities that you have encountered so far – including the Addition Formulae, the Sum and Product Formulae and the Multiple Angle Identities – in order to solve the given equations.

Example

Solve $\cos 5\theta + \cos\theta = \cos 3\theta$ for $0° \le \theta \le 180°$

Solution

Using $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\text{LHS} = 2\cos 3\theta \cos 2\theta$

Thus $2\cos 3\theta\cos 2\theta = \cos 3\theta$

$\Rightarrow \quad \cos 3\theta(2\cos 2\theta - 1) = 0$

Then

(a) $\cos 3\theta = 0$

$\Rightarrow \quad 3\theta = 90°, 270°, 450°$

$\Rightarrow \quad \theta = 30°, 90°, 150°$

or

(b) $2\cos 2\theta - 1 = 0$

$\Rightarrow \quad \cos 2\theta = \tfrac{1}{2}$

$\Rightarrow \quad 2\theta = 60°, 300°$

$\Rightarrow \quad \theta = 30°, 150°$ as already found.

Solutions are $\theta = 30°\text{(twice)}, 90°, 150°\text{(twice)}$.

[Remember, for final solutions in range $0° \le \theta \le 180°$, solutions for 3θ must be in range $0° \le \theta \le 3 \times 180° = 540°$.]

Exercise 1D

1. Solve for $0° \le \theta \le 180°$:

   (a) $\cos\theta + \cos 3\theta = 0$ (b) $\sin 4\theta + \sin 3\theta = 0$

   (c) $\sin\theta + \sin 3\theta = \sin 2\theta$.

2. Find all the values of x satisfying the equation

   $\sin x = 2\sin\left(\frac{\pi}{3} - x\right)$ for $0 \le x \le 2\pi$.

3. By writing 3θ as $2\theta+\theta$, show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

and find a similar expresion for $\sin 3\theta$ in terms of powers of $\sin\theta$ only.

Use these results to solve, for $0 \le \theta \le 360°$,

(a) $\cos 3\theta + 2\cos\theta = 0$

(b) $\sin 3\theta = 3\sin 2\theta$

(c) $\cos\theta - \cos 3\theta = \tan^2\theta$

4. Show that $\tan x + \cot x = 2\operatorname{cosec} 2x$. Hence solve $\tan x + \cot x = 8\cos 2x$ for $0 \le x \le \pi$.

5. Solve the equation

$$\sin 2x + \sin 3x + \sin 5x = 0 \text{ for } 0° \le x \le 180°$$

6. Find the solution x in the range $0° \le x \le 360°$ for which $\sin 4x + \cos 3x = 0$.

7. (a) Given $t = \tan\frac{1}{2}\theta$, write down $\tan\theta$ in terms of t and show that

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

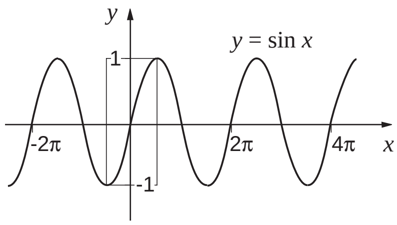
Find also a similar expression for $\sin\theta$ in terms of t.

(b) Show that $2\sin\theta - \tan\theta = \dfrac{2t}{1-t^4}\left(1-3t^2\right)$

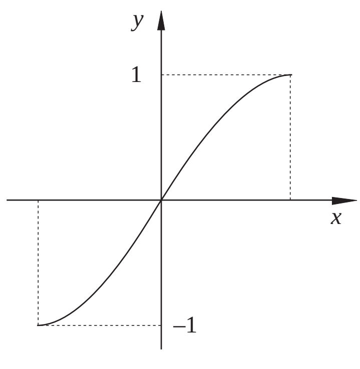
(c) Hence solve $2\sin\theta - \tan\theta = 6\cot\frac{1}{2}\theta$ for values of θ in the range $0° < \theta < 360°$.

1.5 The inverse trigonometric functions

In the strictest sense, for a function f to have an inverse it has to be 1–1 ('one-to-one'). Now the three trigonometric functions sine, cosine and tangent are each periodic. Thus the equation $\sin x = k$, for $|k| \le 1$, has infinitely many solutions x. A sketch of the graph of $y = \sin x$ is shown opposite.



When working on your calculator, if you find $\sin^{-1} 0.5$, say, a single answer is given, despite there being infinitely many to choose from. In order to restrict a 'many-to-one' function of this kind into a 1–1 function, so that the inverse function gives a unique answer , the range of values is restricted. This can be done in a number of ways, but the most sensible way is to choose a range of values x which includes the acute angles. This is shown on the diagram opposite.



Thus for $-1 \le k \le 1$,

$$\sin x = k \quad \Rightarrow \quad x = \sin^{-1} k$$

will be assigned a unique value of x in the range $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$: these are the principal values of the inverse-sine function.

Activity 4

By drawing the graphs $y = \cos x$ and $y = \tan x$, find the ranges of principal values of the inverse-cosine and inverse-tangent functions. (These should include the acute angles of x.)

Note that the inverse functions are denoted here by

$$\sin^{-1} x, \quad \cos^{-1}, \quad \tan^{-1} x$$

These are not the same as

$$(\sin x)^{-1} = \frac{1}{\sin x}, \quad \text{etc.}$$

and to avoid this confusion, some texts denote the inverse functions as

$$arc\sin x, \quad arc\cos x, \quad arc\tan x.$$

1.6 General solutions

Up until now you have been asked for solutions of trigonometric equations within certain ranges. For example:

> Solve $\sin 3x = \frac{1}{2}$ for $0° < x < 180°$

or

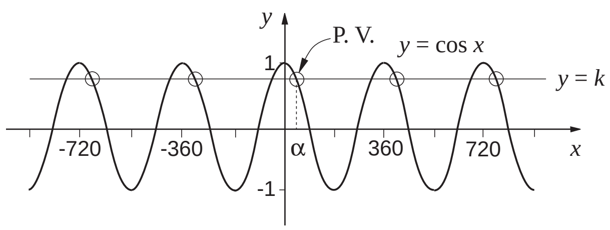
> Find the values of θ for which $\sin 2\theta + \sin^2 \theta = 0$ with $0 \le \theta \le 2\pi$.

At the same time, you will have been aware that even the simplest trig equation can have infinitely many solutions: $\sin\theta = 0$ (θ radians) is true when $\theta = 0, \pi, 2\pi, 3\pi, \ldots$ and **also** for all negative multiples of π as well.

Overall, one could say that the equation $\sin\theta = 0$ has general solution $\theta = n\,\pi$ where *n* is an integer. Moreover, there are no values of θ which satisfy this equation that **do not** take this form. Thus, '$\theta = n\,\pi$' describes **all** the values of θ satisfying '$\sin\theta = 0$' as *n* is allowed to take any integer value. This is what is meant by a **general solutio**n.

General solution for the cosine function

For $-900° \le x \le 900°$ (radians can come later), the graph of $y = \cos x$ is shown below.



The line $y = k$ (k is chosen as positive here, but as long as $-1 \le k \le 1$, the actual value is immaterial) is also drawn on the sketch.

The **principal value** of x for which $x = \cos^{-1} k$ (representing the point when $y = k$ and $y = \cos x$ intersect) is circled and labelled 'P.V.' Since the cosine function is periodic with period $360°$, all other solutions to the equation $\cos x = k$ corresponding to this principal value are obtained by adding, or subtracting, a multiple of $360°$ to it. The points of intersection of the two graphs representing these solutions are circled also.

Now the cosine curve here is symmetric in the y-axis. So if α is the principal value of x for which $\cos x = k$, then $-\alpha$ is also a solution, and this is not obtained by adding, or subtracting, a multiple of $360°$ to, or from, α. All the remaining solutions of the equation can be obtained by adding or subtracting a multiple of $360°$ to or from $-\alpha$.

The general solution of the equation

$$\cos x = k \quad (-1 \le k \le 1)$$

is then

$$x = 360n \pm \alpha$$

where

$$\alpha = \cos^{-1} k$$

is the principal value of the inverse cosine function and n is an integer.

In radians, using $360° \equiv 2\pi$ radians, the general solution looks like

$$x = 2n\pi \pm \alpha, \; n \text{ an integer.}$$

Activity 5

Use the graphs of $y = \tan x$ and $y = \sin x$ to find the general solutions of the equations (in degrees) of the equations

$$\tan x = k \qquad (-\infty < k < \infty)$$

and

$$\sin x = k \qquad (-1 \le k \le 1).$$

In each case, let α be the principal value concerned, let n be an integer, and express the general solutions in terms of radians once the results have been found in terms of degrees.

These results are summarised as follows:

In radians	**In degrees**
If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n \alpha$	If $\sin\theta = \sin\alpha$, then $\theta = 180n + (-1)^n \alpha$
If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$	If $\cos\theta = \cos\alpha$, then $\theta = 360n \pm \alpha$
If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$	If $\tan\theta = \tan\alpha$, then $\theta = 180n + \alpha$

[In each case, n is an integer.]

The AEB's *Booklet of Formulae* gives only the set of results for θ, α in radians, but you should be able to convert the results into degrees without any difficulty by remembering that $180° \equiv \pi$ radians, etc.

Example

Find the general solution, in degrees, of the equation

$$\tan 3\theta = \sqrt{3}$$

Solution

$$\tan 3\theta = \sqrt{3}$$

$$\Rightarrow \quad \tan 3\theta = \tan 60°$$

$$\Rightarrow \quad 3\theta = 180n + 60 \text{ quoting the above result}$$

$$\Rightarrow \quad \theta = (60n + 20)°$$

$$[\ldots; n = -1 \Rightarrow \theta = -40°; n = 0 \Rightarrow \theta = 20°; n = 1 \Rightarrow 0 = 80°; \ldots]$$

Sometimes, you may have to do some work first.

Example

Find the general solution, in radians, of the equation

$$8\sin\theta + 15\cos\theta = 6.$$

Solution

Rewriting the LHS of this equation in the form $R\sin(\theta+\alpha)$, for instance, gives $R=\sqrt{8^2+15^2}=17$ and $\cos\theta=\dfrac{8}{17}$ or $\sin\theta=\dfrac{15}{17}$ or $\tan\theta=\dfrac{15}{8}$ so that $\theta\approx 1.081$ radians. [Check this working through to make sure you can see where it comes from.]

The equation can now be written as

$$17\sin(\theta+1.081)=6$$

$$\Rightarrow \quad \sin(\theta+1.081)=\frac{6}{17}=\sin 0.3607 \qquad \text{(principal value of } \sin^{-1}\tfrac{6}{17}\text{)}$$

$$\Rightarrow \quad \theta+1.081=n\pi+(-1)^n\,0.3607$$

$$\Rightarrow \quad \theta=n\pi+(-1)^n\,0.3607-1.081$$

One could proceed to make this more appealing to the eye by considering the cases *n* even and *n* odd separately, but there is little else to be gained by proceeding in this way.

Note on accuracy: although final, non-exact numerical answers are usually required to three significant places, the 0.3607 and 1.081 in the answer above are really intermediate answers and hence are given to 4 significant figure accuracy. However, unless a specific value of *n* is to be substituted in order to determine an individual value of θ, you will not be penalised for premature rounding provided your working is clear and the answers correspond appropriately.

This final example illustrates the sort of ingenuity you might have to employ in finding a general solution of some equation.

Example

Find the values of *x* for which $\cos x-\sin 4x=0$.

Solution

Using the result $\sin A=\cos\left(\dfrac{\pi}{2}-A\right)$, the above equation can be written as

$$\cos x=\cos\left(\frac{\pi}{2}-4x\right)$$

whence

$$x=2n\pi\pm\left(\frac{\pi}{2}-4x\right)$$

i.e.

$$x = 2n\pi + \frac{\pi}{2} - 4x \quad \text{or} \quad x = 2n\pi - \frac{\pi}{2} + 4x$$

$$\Rightarrow \quad 5x = 2n\pi + \frac{\pi}{2} \quad \text{or} \quad 3x = -2n\pi + \frac{\pi}{2}$$

$$\Rightarrow \quad x = 2n\frac{\pi}{5} + \frac{\pi}{10} \quad \text{or} \quad x = -2n\frac{\pi}{3} + \frac{\pi}{6}$$

Wait a moment! Although the second solution is correct, n is merely an indicator of some integer; positive, negative or zero. It is immaterial then, whether it is denoted as positive or negative: you could write this solution as $x = 2k\frac{\pi}{3} + \frac{\pi}{6}$ $(k = -n)$ for some integer k, or alternatively, as $x = 2n\frac{\pi}{3} + \frac{\pi}{6}$.

In this case, the general solution takes two forms. An alternative approach could have re-written the equation as

$$\sin 4x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\Rightarrow \quad 4x = n\pi + (-1)^n\left(\frac{\pi}{2} - x\right)$$

When n is odd

$$4x = n\pi - \frac{\pi}{2} + x$$

$$\Rightarrow \quad 3x = n\pi - \frac{\pi}{2}$$

$$\Rightarrow \quad x = n\frac{\pi}{3} - \frac{\pi}{6} = (2n-1)\frac{\pi}{6} \qquad (n \text{ odd})$$

and when n is even

$$4x = n\pi + \frac{\pi}{2} - x$$

$$\Rightarrow \quad 5x = n\pi + \frac{\pi}{2}$$

$$\Rightarrow \quad x = n\frac{\pi}{5} + \frac{\pi}{10} = (2n+1)\frac{\pi}{10} \qquad (n \text{ even})$$

The very first way might be considered preferable since the general solution for cos is less clumsy than that for sin.

This example also highlights another important point: two equivalent sets of answers may look **very** different from each other and yet still both be correct.

Exercise 1E

1. Find the general solutions, in degrees, of the equations

   (a) $\sin x = 0.766$ (b) $\tan(\theta - 45°) = \frac{1}{\sqrt{3}}$

   (c) $\cos x = 0.17$ (d) $\cot(60° - 2\theta) = 3$

   (e) $5\sin x + 3\cos x = 4$ (f) $4\cos\theta + 3\sin\theta = 2$

   (g) $\cos 3\theta + \cos\theta = 0$ (h) $\tan^2 4x = 3$

   (i) $\sin 7x - \sin x = \cos 4x$

2. Find the general solutions, in radians, of the equations

   (a) $\tan x = \sqrt{2}$ (b) $\cos\left(2x + \frac{\pi}{6}\right) = 1$

   (c) $\sin x = 0.35$ (d) $\sec\left(\frac{1}{2}x + \frac{\pi}{4}\right) = 2$

   (e) $\sqrt{6}\sin\theta - \sqrt{2}\cos\theta = 2$

   (f) $10\cos\theta - 24\sin\theta = 13$

   (g) $\cos x - \cos 3x = 0$

   (h) $\tan x + \cot 2x = 0$

   [Note: $\cot A = \tan\left(\frac{\pi}{2} - A\right)$ and $\tan(-A) = -\tan A$]

   (i) $3\tan^2\theta + 5\sec\theta + 1 = 0$

   (j) $\cos 4x + \cos 6x = \cos 5x$

3. Prove the identity $\cos 4x + 4\cos 2x = 8\cos^4 x - 3$. Hence find the general solution, in radians, of the equation $2\cos 4\theta + 8\cos 2\theta = 3$.

1.7 Calculus of the inverse trigonometric functions

The prospect of having to differentiate the function $y = \sin^{-1} x$ may seem rather daunting. However, we can write $y = \sin^{-1} x$ as $\sin y = x$. (Taking the principal range values $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ ensures that this can be done.) Then, using the Chain Rule for differentiation,

$$\frac{d}{dx}(\sin y) = \frac{d}{dy}(\sin y)\frac{dy}{dx} = \cos y \frac{dy}{dx}$$

so that $\sin y = x$ differentiates to give

$$\Rightarrow \quad \cos y \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

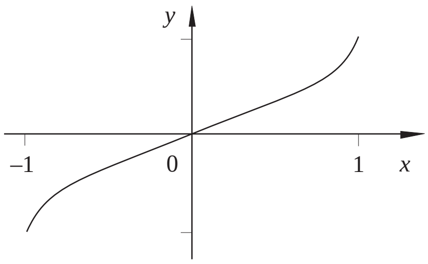
Now, using

$$\cos^2 y = 1 - \sin^2 y,$$

so that $$\cos^2 y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}$$

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1-x^2}}$$

A quick look at the graph of $\sin^{-1} x$ shows that the gradient of the inverse-sine curve is always positive (technically speaking, infinitely so at $x = \pm 1$) and so $y = \sin^{-1} x$ differentiates to



$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Activity 6

Use the above approach to find $\frac{dy}{dx}$ when $y = \tan^{-1} x$.

Find the derivative of $\cos^{-1} x$ also, and decide why it is not necessary to learn this result as well as the result for the derivative of $\sin^{-1} x$ when reversing the process and integrating.

The results

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

and the corresponding integrations

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C \qquad (C \text{ constant})$$

are special cases of the more general results:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \quad \text{for constant } C.$$

The results, in this form, are given in the AEB's *Booklet of Formulae*, and may be quoted when needed.

Activity 7

(a) Use the substitution $x = a\sin\theta$ to prove the result

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

(b) Use the substitution $x = a\tan\theta$ to prove the result

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

Example

Evaluate $\displaystyle\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1 - 4x^2}}$

Solution

Now $\sqrt{1 - 4x^2} = \sqrt{4}\sqrt{\frac{1}{4} - x^2} = 2\sqrt{\frac{1}{4} - x^2}$

so that $\displaystyle\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1 - 4x^2}} = \frac{1}{2}\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{\frac{1}{4} - x^2}}$ (and this is now the standard format, with $a = \frac{1}{2}$)

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x}{\frac{1}{2}}\right)\right]_0^{\frac{1}{4}}$$

$$= \frac{1}{2}\left[\sin^{-1} 2x\right]_0^{\frac{1}{4}}$$

$$= \frac{1}{2}\left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6} - 0\right)$$

$$= \frac{\pi}{12}$$

Example

Evaluate $\int_1^2 \frac{3dx}{2+x^2}$,

giving your answer correct to 4 decimal places.

Solution

$$\int_1^2 \frac{3dx}{2+x^2} = 3\int_1^2 \frac{dx}{\left(\sqrt{2}\right)^2 + x^2} \qquad [\text{so } a = \sqrt{2} \text{ here}]$$

$$= 3\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_1^2$$

$$= \frac{3}{\sqrt{2}}\left(\tan^{-1}\sqrt{2} - \tan^{-1}\frac{1}{\sqrt{2}}\right) \qquad [\text{Important note: work in radians}]$$

$$\approx \frac{3}{\sqrt{2}}(0.95532 - 0.61548)$$

$$= 0.7209 \qquad (4 \text{ d.p.})$$

Exercise 1F

Evaluate the following integrals, giving your answers to four significant figures. [Remember to work in radians.]

1. $\int_{\frac{1}{4}}^{\frac{1}{8}} \frac{dx}{\sqrt{1-8x^2}}$

2. $\int_4^8 \frac{4}{8+x^2}\,dx$

3. $\int_2^{\sqrt{5}} \frac{dx}{5+9x^2}$

4. $\int_0^2 \frac{7}{\sqrt{7-x^2}}\,dx$

Evaluate the following integrals exactly:

5. $\int_0^{\sqrt{3}} \frac{3}{6+2x^2}\,dx$

6. $\int_{\frac{3}{2}}^{\sqrt{3}} \frac{2dx}{\sqrt{3-x^2}}$

7. $\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{6+x^2}$

8. $\int_0^{\frac{\sqrt{5}}{2}} \frac{dx}{\sqrt{5-2x^2}}$

9. $\int_0^1 \frac{dx}{\sqrt{4-3x^2}}$

*10. $\int_{-1}^1 \left(\frac{1-x^2}{1+x^2}\right) dx$

1.8 The '$t = \tan\frac{1}{2}A$' substitution

You will have already encountered the result:

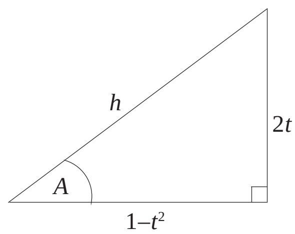
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

which arises from the identity

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Setting $\theta = \frac{1}{2}A$ then yields the result

$$\tan A = \frac{2t}{1-t^2} \text{ where } t = \tan\frac{1}{2}A$$



In the triangle shown opposite, $\tan A = \frac{2t}{1-t^2}$, and the hypotenuse, h, is given by Pythagoras' theorem:

$$\begin{aligned} h^2 &= \left(1-t^2\right) + (2t)^2 \\ &= 1 - 2t^2 + t^4 + 4t^2 \\ &= 1 + 2t^2 + t^4 \\ &= \left(1+t^2\right)^2 \end{aligned}$$

So $h = 1 + t^2$ and

$$\sin A = \frac{2t}{1+t^2} \text{ and } \cos A = \frac{1-t^2}{1+t^2}$$

This would seem, at first sight, to be merely an exercise in trigonometry manipulation, but these results (which are given in the AEB's *Booklet of Formulae*) have their uses, particularly in handling some otherwise tricky trigonometric integrations.

Incidentally, the above working pre-supposes that angle A is acute, and this is generally the case in practice. The identities are valid, however, for all values of A in the range $0 \le A \le 2\pi$ (and hence all values of A).

Although, when $\frac{1}{2}A = \frac{\pi}{2}, 3\frac{\pi}{2}, \ldots \tan\frac{1}{2}A$ is not defined, the limiting values of sin A, cos A and tan A are still correct.

Example

Use the substitution $t = \tan\frac{1}{2}x$ to show that the indefinite integral of sec x is $\ln\left|\tan\left(\frac{\pi}{4}+\frac{1}{2}x\right)\right|$.

Solution

$$t = \tan\frac{1}{2}x \quad \Rightarrow \quad \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{1}{2}x$$

$$\Rightarrow \quad 2dt = \left(\sec^2\frac{1}{2}x\right)dx$$

$$\Rightarrow \quad 2dt = \left(1+\tan^2\frac{1}{2}x\right)dx$$

$$\Rightarrow \quad \frac{2dt}{1+t^2} = dx$$

Also, $\sec x = \frac{1}{\cos x} = \frac{1+t^2}{1-t^2}$, using one of the above results.

Then

$$\int \sec x\,dx = \int \frac{1+t^2}{1-t^2}.\frac{2dt}{1+t^2}$$

$$= \int \frac{2}{1-t^2}\,dt$$

$$= \int \frac{2}{(1-t)(1+t)}\,dt$$

$$= \int \left\{\frac{1}{1+t}+\frac{1}{1-t}\right\}dt \quad \text{by partial fractions}$$

$$= \ln|1+t| - \ln|1-t|$$

$$= \ln\left|\frac{1+t}{1-t}\right|$$

Now $$\tan\left(\frac{\pi}{4}+\frac{1}{2}x\right) = \frac{\tan\frac{\pi}{4}+\tan\frac{1}{2}x}{1-\tan\frac{\pi}{4}-\tan\frac{1}{2}x} = \frac{1+t}{1-t} \qquad \left(\text{since } \tan\frac{\pi}{4}=1\right)$$

so that

$$\int \sec x\,dx = \ln\left|\tan\left(\frac{\pi}{4}+\frac{1}{2}x\right)\right|$$

Activity 8

Use the above identities for cos x and tan x to prove that

$$\sec x + \tan x = \frac{1 + \tan \frac{1}{2} x}{1 - \tan \frac{1}{2} x}$$

The results $\sec x\, dx = \ln\left| \sec x + \tan x \right| = \ln\left| \tan\left(\frac{\pi}{4} + \frac{1}{2} x \right) \right|$ are given in the AEB's *Booklet of Formulae*, as is the result

$$\int \operatorname{cosec} x\, dx = \ln\left| \tan \frac{1}{2} x \right|$$

Since this latter result is much easier to establish, it has been set as an exercise below.

Example

Use the substitution $t = \tan \frac{1}{2} \theta$ to evaluate exactly

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\cos\theta + 3\sin\theta}$$

Solution

$$t = \tan \frac{1}{2} \theta \Rightarrow \frac{dt}{d\theta} = \frac{1}{2} \sec^2 \frac{1}{2} \theta \Rightarrow \frac{2dt}{1+t^2} = d\theta$$

Also,

$$\cos\theta = \frac{1-t^2}{1+t^2} \text{ and } \sin\theta = \frac{2t}{1+t^2}$$

Changing the limits:

$$\theta = 0 \Rightarrow t = 0 \text{ and } \theta = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{4} = 1$$

so

$$\left(\theta, \frac{\pi}{2} \right) \to (0,1)$$

Thus

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\cos\theta + 3\sin\theta} = \int_0^1 \frac{1}{\frac{4-4t^2}{1+t^2} + \frac{6t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2\,dt}{4 - 4t^2 + 6t}$$

$$= \int_0^1 \frac{dt}{2 + 3t - 2t^2}$$

$$= \int_0^1 \left\{ \frac{\frac{1}{5}}{2 - t} + \frac{\frac{2}{5}}{1 + 2t} \right\} dt$$

$$= \left[-\frac{1}{5}\ln|2 - t| + \frac{1}{5}\ln|1 + t| \right]_0^1$$

$$= \left(-\frac{1}{5}\ln 1 + \frac{1}{5}\ln 3 \right) - \left(-\frac{1}{5}\ln 2 - \frac{1}{5}\ln 1 \right)$$

$$= \frac{1}{5}\ln 6$$

In the following exercise, you may have to use the $\tan^{-1}$ or $\sin^{-1}$ integrals from the previous section.

Exercise 1G

1. Use the substitution $t = \tan\frac{1}{2}x$ to evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{1 + \sin^2 x}$, giving your answer to 4 decimal places.

2. By writing $t = \tan\frac{1}{2}x$, show that

   (a) $\int \operatorname{cosec} x\, dx = \ln\left|\tan\frac{1}{2}x\right| + C$

   (b) $\int \frac{dx}{1 + \cos x} = \tan\frac{1}{2}x + C$

   (c) $\int \frac{dx}{\cos\frac{1}{2}x\sqrt{\cos x}} = 2\sin^{-1}\left\{\tan\frac{1}{2}x\right\} + C$

3. Use the $t = \tan\frac{1}{2}\theta$ substitution to evaluate exactly the integrals

   (a) $\int_0^{\frac{2\pi}{3}} \frac{d\theta}{5 + 4\cos\theta}$ (b) $\int_0^{\frac{\pi}{2}} \frac{\tan\frac{1}{2}\theta}{5 + 4\cos\theta}\, d\theta$

   (c) $\int_0^{\frac{\pi}{2}} \frac{1}{3 + 5\sin\theta}\, d\theta$

4. (a) By using the identity $\tan A = \frac{2t}{1 - t^2}$ where $t = \tan\frac{1}{2}A$, and setting $A = \frac{\pi}{6}$, show that

   $$\tan\frac{\pi}{12} = 2 - \sqrt{3}$$

   (b) Evaluate, to four decimal places, the integral

   $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{d\theta}{2 + \cos\theta}$$

5. By setting $t = \tan\frac{1}{2}x$, find the indefinite integral

   $$\int \sec\frac{1}{2}x\sqrt{1 - \cos x}\, dx$$

1.9 Harder integrations

In this final section of the chapter, all of the integrations involve the standard results for $\sin^{-1}$ and $\tan^{-1}$, but you may have to do some work to get them into the appropriate form.

Before you start, here are a few reminders of the algebraic techniques which you will need, and also one or two calculus results. To give you a clear idea of how they work out in practice, they are incorporated into the following set of examples.

Example

By writing $\dfrac{x^2+7x+2}{(1+x^2)(2-x)}$ in terms of partial fractions, show that

$$\int_0^1 \frac{x^2+7x+2}{(1+x^2)(2-x)}\,dx = \frac{\pi}{2}\ln 2 - \frac{\pi}{4}$$

Solution

$$\frac{x^2+7x+2}{(1+x^2)(2-x)} \equiv \frac{Ax+B}{1+x^2} + \frac{C}{2-x}$$

Multiplying throughout by the denominator

$$x^2+7x+2 \equiv (Ax+B)(2-x) + C(1+x^2)$$

Substituting $x=2$ gives

$$20 = 5C \quad \Rightarrow \quad C = 4$$

Substituting $x=0$ gives

$$2 = 2B+4 \quad \Rightarrow \quad B = -1$$

Comparing x^2 coefficients:

$$1 = -A+4 \quad \Rightarrow \quad A = 3$$

Thus $$\frac{x^2+7x+2}{(1+x^2)(2-x)} \equiv \frac{3x-1}{1+x^2} + \frac{4}{2-x}$$

Then $$\int_0^1 \frac{x^2+7x+2}{(1+x^2)(2-x)}\,dx = \int_0^1 \frac{3x}{1+x^2}\,dx - \int_0^1 \frac{1}{1+x^2}\,dx + 4\int_0^1 \frac{1}{2-x}\,dx$$

The reason for splitting the integration up in this way is to separate the directly integrable bits. Note that

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + \text{constant}$$

i.e. when the 'top' is exactly the differential of the 'bottom', the integral is natural log of the 'bottom'. Now $\frac{d}{dx}\left(1+x^2\right) = 2x$ and $\frac{d}{dx}(2-x) = -1$, so the constants in the numerators need jiggling but, apart from this, you should see that the three integrals are log, $\tan^{-1}$ and log respectively:

$$= \frac{3}{2}\int_0^1 \frac{2x}{1+x^2}\,dx - \int_0^1 \frac{1}{1+x^2}\,dx - 4\int_0^1 \frac{-1}{2-x}dx$$

$$= \frac{3}{2}\left[\ln\left(x^2+1\right)\right]_0^1 - \left[\tan^{-1} x\right]_0^1 - 4\left[\ln(2-x)\right]_0^1$$

[Strictly speaking, the log integrals should be $|x^2+1|$ and $|2-x|$, but $1+x^2$ is always positive and $2-x$ is positive for x between 0 and 1.]

$$= \frac{3}{2}(\ln 2 - \ln 1) - \left(\tan^{-1} 1 - \tan^{-1} 0\right) - 4(\ln 1 - \ln 2)$$

$$= \frac{3}{2}\ln 2 - \frac{\pi}{4} + 4\ln 2$$

$$= \frac{11}{2}\ln 2 - \frac{\pi}{4}$$

Example

Evaluate $\int_0^{\frac{1}{3}} \frac{3x+1}{\sqrt{1-3x^2}}dx$, giving your answer correct to three significant figures.

Solution

$$\int_0^{\frac{1}{3}} \frac{3x+1}{\sqrt{1-3x^2}}\,dx = \int_0^{\frac{1}{3}} \frac{3x}{\sqrt{1-3x^2}}\,dx + \int_0^{\frac{1}{3}} \frac{1}{\sqrt{1-3x^2}}\,dx$$

Now the second integral on the RHS is clearly a $\sin^{-1}$ integral.

What about the first one?

You might just recognise where the x in the numerator comes from if you think about it long enough. To save time: try calling

$$u = \sqrt{1-3x^2} = \left(1-3x^2\right)^{\frac{1}{2}};$$

then $$\frac{du}{dx} = \frac{1}{2}\left(1-3x^2\right)^{-\frac{1}{2}}(-6x) \quad \text{using the Chain Rule}$$

$$= \frac{-3x}{\sqrt{1-3x^2}}$$

So $$\int \frac{3x}{\sqrt{1-3x^2}}\,dx = -\sqrt{1-3x^2}$$

[This method is referred to as 'integration by recognition']

and

$$\int_0^{\frac{1}{3}} \frac{3x+1}{\sqrt{1-3x^2}}dx = \int_0^{\frac{1}{3}} \frac{3x}{\sqrt{1-3x^2}}\,dx + \frac{1}{\sqrt{3}}\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{\frac{1}{3}-x^2}}$$

$$= \left[-\sqrt{1-3x^2}\right]_0^{\frac{1}{3}} + \frac{1}{\sqrt{3}}\left[\sin^{-1}\left(\frac{x}{\frac{1}{\sqrt{3}}}\right)\right]_0^{\frac{1}{3}}$$

$$= \left(-\sqrt{\frac{2}{3}}+1\right) + \frac{1}{\sqrt{3}}\left(\sin^{-1}\frac{1}{\sqrt{3}} - \sin^{-1}0\right)$$

$$= 0.539 \text{ (to 3 s.f.)}$$

Example

Integrate exactly the integrals

(a) $\displaystyle\int_1^2 \frac{dx}{3x^2-6x+4}$ (b) $\displaystyle\int_1^{\frac{3}{2}} \frac{dx}{\sqrt{-2+5x-2x^2}}$

Solution

(a) $$\int_1^2 \frac{dx}{3x^2-6x+4} = \frac{1}{3}\int_1^2 \frac{dx}{x^2-2x+\frac{4}{3}}$$

[Now $x^2-2x+\frac{4}{3} = (x-1)^2+\frac{1}{3}$ by completing the square: the factor of 3 was taken out first in order to make this easier to cope with.]

$$= \frac{1}{3}\int_1^2 \frac{dx}{\frac{1}{3}+(x-1)^2}$$

[This is $\displaystyle\int \frac{dx}{a^2+x^2}$ with $a = \frac{1}{\sqrt{3}}$ and 'x' $= x-1$, which is allowed when a single x is involved.]

$$= \frac{1}{3}\left[\frac{1}{\frac{1}{\sqrt{3}}}\tan^{-1}\left(\frac{x-1}{\frac{1}{\sqrt{3}}}\right)\right]_1^2$$

$$= \frac{\sqrt{3}}{3}\left[\tan^{-1}\left([x-1]\sqrt{3}\right)\right]_1^2$$

$$= \frac{\sqrt{3}}{3}\left(\tan^{-1}\sqrt{3} - \tan^{-1} 0\right)$$

$$= \frac{\sqrt{3}}{3}\left(\frac{\pi}{3} - 0\right)$$

$$= \frac{\pi\sqrt{3}}{9}$$

(b) $\displaystyle\int_1^{\frac{3}{2}} \frac{dx}{\sqrt{-2+5x-2x^2}} = \frac{1}{\sqrt{2}}\int_1^{\frac{3}{2}} \frac{dx}{\sqrt{-1+\frac{5}{2}x-x^2}}$

Completing the square:

$$-1+\frac{5}{2}x-x^2 = -\left\{x^2-\frac{5}{2}x+1\right\} = -\left\{\left(x-\frac{5}{4}\right)^2 - \frac{25}{16}+1\right\}$$

$$= \frac{9}{16} - \left(x-\frac{5}{4}\right)^2$$

This gives

$$\text{integral} = \frac{1}{\sqrt{2}}\int_1^{\frac{3}{2}} \frac{dx}{\sqrt{\left(\frac{3}{4}\right)^2 - \left(x-\frac{5}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\left(\frac{\left(x-\frac{5}{4}\right)}{\frac{3}{4}}\right)\right]_1^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\left(\frac{(4x-5)}{3}\right)\right]_1^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{2}}\left(\sin^{-1}\frac{1}{3} - \sin^{-1}\left(-\frac{1}{3}\right)\right)$$

$$= \frac{1}{\sqrt{2}}\left(2\sin^{-1}\frac{1}{3}\right)$$

$$= \sqrt{2}\sin^{-1}\frac{1}{3}$$

Exercise 1H

1. By expressing $\frac{3x+10}{(2-x)(4+x^2)}$ in terms of partial fractions, evaluate

$$\int_0^1 \frac{3x+10}{(2-x)(4+x^2)}dx$$

giving your answer to three significant figures.

2. Show that $\int_0^{\sqrt{\frac{3}{2}}} \frac{3x+2}{9+2x^2}dx = \frac{3}{4}\ln\left(\frac{4}{3}\right)+\frac{\pi\sqrt{2}}{18}$.

3. Given that $y=\sqrt{4-x^2}$, find an expression for $\frac{dy}{dx}$ and deduce that $\int \frac{x}{\sqrt{4-x^2}}dx = C-\sqrt{4-x^2}$ for some constant C. Hence evaluate exactly $\int_1^2 \frac{2x-3}{\sqrt{4-x^2}}dx$

4. Determine the values of the constants A, B and C such that

$$\frac{x^2+2x+7}{1+x^2} \equiv A+\frac{Bx}{1+x^2}+\frac{C}{1+x^2}$$

Show that $\int_1^{\sqrt{3}} \frac{x^2+2x+7}{1+x^2}dx = \sqrt{3}-1+\ln 2+\frac{\pi}{2}$

5. By 'completing the square' in each of the following cases, evaluate exactly the integrals:

(a) $\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dx}{\sqrt{2x-x^2}}$ (b) $\int_0^1 \frac{dx}{x^2-x+1}$

(c) $\int_0^{\frac{2}{3}} \frac{dx}{\sqrt{1+2x-3x^2}}$ (d) $2x^2-2x+5$

(e) $\int_0^1 \frac{dx}{\sqrt{x-x^2}}$ (f) $\int_0^1 \frac{dx}{x^2+6x+10}$

6. Show that $\int_0^2 \frac{4-3x}{4+3x^2}dx = \frac{2\pi}{3\sqrt{3}}-\frac{1}{2}\ln 13$.

7. Use partial fractions to help evaluate the integral

$$\int_0^{\sqrt{3}} \frac{5}{(1+x^2)(2+x)}dx$$

8. Evaluate the following integrals:

(a) $\int_1^4 \frac{5x+4}{x^2+4}dx$ (b) $\int_0^{\sqrt{2}} \frac{4+3x-x^2}{2+x^2}dx$

9. Determine the values of the constants A, B and C for which

$$f(x)=\frac{x^2+2x-4}{x^2-2x+4} \equiv A+B\left(\frac{2x-2}{x^2-2x+4}\right)+\frac{C}{x^2-2x+4}$$

Hence evaluate $\int_1^4 f(x)$.

10. (a) Prove that $\int_1^2 \frac{4x}{5+x^2}dx = 2\ln\left(\frac{3}{2}\right)$.

(b) Use the result of (a) to evaluate $\int_1^2 \frac{4x+2}{5+x^2}dx$, giving your answer correct to 3 decimal places.

*11. Show that $\int_0^1 \frac{3x\,dx}{\sqrt{1+6x-3x^2}} = \frac{\pi}{\sqrt{3}}-1$.

*12. Show that $\int_0^{\frac{\sqrt{3}}{2}} \frac{1+x^3}{\sqrt{1-x^2}}dx = \frac{1}{24}(8\pi+5)$.

[Hint: $\int \frac{x^3}{\sqrt{1-x^2}}dx = \int x^2 \times \frac{x}{\sqrt{1-x^2}}dx$]

1.10 Miscellaneous Exercises

1. Prove the identity $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$. Find, in radians, all the solutions of the equation $\tan x + \cot x = 8\cos 2x$ in the interval $0 < x < \pi$. (AEB)

2. Find, in radians, the general solution of the equation $6\tan^2\theta = 4\sin^2\theta + 1$. (AEB)

3. Prove the identity $\cot 2\theta + \tan\theta \equiv \operatorname{cosec} 2\theta$. Hence find the values of θ, in the interval $0° < \theta < 180°$ for which $3(\cot 2\theta + \tan\theta)^2 = 4$. (AEB)

4. Find, in terms of π, the general solution of the equation $\tan^4 x - 4\tan^2 x + 3 = 0$. (AEB)

5. Solve the equation $\sqrt{3}\tan\theta - \sec\theta = 1$, giving all solutions in the interval $0° < \theta < 360°$. (AEB)

6. By expanding $\cos(\theta - 60°)$, express $7\cos\theta + 8\cos(\theta - 60°)$ in the form $13\sin(\theta + \alpha)$, where $0° < \alpha < 90°$, and state the value of α to the nearest $0.1°$. Hence find the solutions of the equation $7\cos\theta + 8\cos(\theta - 60°) = 6.5$ in the interval $0° < \theta < 360°$, giving your answer to the nearest $0.1°$. (AEB)

7. Find all solutions in the interval $0° \le \theta \le 360°$ of the equation $\sin\theta - \cos\theta = k$ when

   (a) $k = 0$, and (b) $k = 1$. (AEB)

8. Find the general solution of the equation $\sin 2x + 2\cos^2 x = 0$ for x radians. (AEB)

9. Show that

   $\sin 2x + \sin 4x + \sin 6x \equiv \sin 4x(1 + 2\cos 2x)$

   Hence prove the identity

   $$\sin 3x \sin 4x \equiv (\sin 2x + \sin 4x + \sin 6x)\sin x$$

   Deduce that $\sin\left(\frac{\pi}{12}\right) = \frac{1}{\sqrt{6} + \sqrt{2}}$. (AEB)

10. Prove the identity

    $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 \equiv 2 + 2\cos(A - B)$.

    Hence solve the equation

    $(\cos 4\theta + \cos\theta)^2 + (\sin 4\theta + \sin\theta)^2 = 2\sqrt{3}\sin 3\theta$

    giving the general solution in degrees.

11. Given that $-1 < x, y < 1$, prove that

    $$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$$

    Deduce the value of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$

    (AEB)

12. Express $f(x) = \frac{3 + x}{(1 + x^2)(1 + 2x)}$ in partial fractions. Prove that the area of the region enclosed by the curve with equation $y = f(x)$, the coordinate axes and the line $x = 1$ is $\frac{\pi}{4} + \frac{1}{2}\ln\left(\frac{9}{2}\right)$.

    (AEB)

13. Express $5\cos\theta + 2\sin\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. The function f is defined by $f(\theta) = 6 - 5\cos\theta - 2\sin\theta$ for $0° \le \theta \le 360°$. State the greatest and least values of f and the values of θ, correct to the nearest $0.1°$, at which these occur.

14. Show that $\int_0^1 \frac{x\,dx}{1 + x^2} = \frac{1}{2}\ln 2$. Hence using integration by parts, evaluate $\int_0^1 \tan^{-1} x\,dx$. (AEB)

15. Express $\frac{16 - x}{(2 - x)(3 + x^2)}$ in partial fractions.

    Hence show that

    $$\int_0^1 \frac{16 - x}{(2 - x)(3 + x^2)}dx = \ln\frac{16}{3} + \frac{5\pi}{6\sqrt{3}}$$ (AEB)

16. Express $\frac{\sin 3\theta}{\sin\theta}$ in terms of $\cos\theta$. Hence show that if $\sin 3\theta = \lambda\sin 2\theta$, where λ is a constant, then either $\sin\theta = 0$ or $4\cos^2\theta - 2\lambda\cos\theta - 1 = 0$. Determine the general solution, in degrees, of the equation $\sin 3\theta = 3\sin 2\theta$. (AEB)

17. Express $3\cos x - 4\sin x$ in the form $A\cos(x + \alpha)$, where $A > 0$ and α is acute, stating the value of α to the nearest $0.1°$.

    (a) Given that $f(x) = \frac{24}{3\cos x - 4\sin x + 7}$:

    (i) Write down the greatest and least values of $f(x)$ and the values of x to the nearest $0.1°$ in the interval $-180° < x < 180°$ at which these occur;

    (ii) Find the general solution, in degrees, of the equation $f(x) = \frac{16}{3}$.

    (b) Solve the equation $3\cos x - 4\sin x = 5\cos 3x$, giving your answers to the nearest $0.1°$ in the interval $0° < x < 180°$. (Oxford)

18. Given that $3\cos\theta + 4\sin\theta \equiv R\cos(\theta - \alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$, state the value of R and the value of $\tan\alpha$.
   (a) For each of the following equations, solve for θ in the interval $0 \le \theta \le 2\pi$ and give your answers in radians correct to one decimal place:
   (i) $3\cos\theta + 4\sin\theta = 2$
   (ii) $3\cos 2\theta + 4\sin 2\theta = 5\cos\theta$.
   (b) The curve with equation $y = \dfrac{10}{3\cos x + 4\sin x + 7}$, between $x = -\pi$ and $x = \pi$, cuts the *y*-axis at *A*, has a maximum point at *B* and a minimum point at *C*. Find the coordinates of *A*, *B* and *C*. (AEB)

19. Given that $f(x) = 9\sin\left(x + \frac{\pi}{6}\right) + 5\cos\left(x + \frac{\pi}{3}\right)$, use the formulae for $\sin(A+B)$ and $\cos(A+B)$ to express *f*(*x*) in the form $C\cos x + D\sqrt{3}\sin x$, where *C* and *D* are integers. Hence show that *f*(*x*) can be written in the form $\sqrt{61}\cos(x - \alpha)$ giving a value for α in radians to three significant figures. (Oxford)

20. Prove the identity $1 + \sin 2\theta \equiv \dfrac{(1 + \tan\theta)^2}{1 + \tan^2\theta}$. By using the substitution $t = \tan\theta$, or otherwise, find the general solution, in radians, of the equation
   $(2 - \tan\theta)(1 + \sin 2\theta) = 2$. (AEB)

21. (a) Starting from the identity
   $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$,
   prove the identity
   $\cos 2\theta \equiv 2\cos^2\theta - 1$.
   (b) Find the general solution of the equation $\sin\theta + \tan\theta\cos 2\theta = 0$, giving your answer in radians in terms of π.
   (c) Prove the identity
   $2\cos^2\theta - 2\cos^2 2\theta \equiv \cos 2\theta - \cos 4\theta$.
   (d) By substituting $\theta = \frac{\pi}{5}$ in the identity in (c), prove that $\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{2\pi}{5}\right) = \frac{1}{2}$.
   (e) Hence find the value of $\cos\left(\frac{\pi}{5}\right)$ in the form $a + b\sqrt{5}$, stating the values of *a* and *b*. (AEB)

22. Use identities for $\cos(C+D)$ and $\cos(C-D)$ to prove that $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.
   Hence find, in terms of π, the general solution of the equation $\cos 5\theta + \cos\theta = \cos 3\theta$.
   Using both the identity for $\cos A + \cos B$, and the corresponding identity for $\sin A - \sin B$, show that
   $\cos 5\alpha - \sin\alpha = 2\sin\alpha(\cos 4\alpha + \cos 2\alpha)$.
   The triangle *PQR* has angle $QPR = \alpha (\ne 0)$, angle $PQR = 5\alpha$ and $RP = 3RQ$. Show that $\sin 5\alpha = 3\sin\alpha$ and deduce that $\cos 4\alpha + \cos 2\alpha = 1$. By solving a quadratic equation in $\cos 2\alpha$, or otherwise, find the value of α, giving your answer to the nearest $0.1°$. (AEB)

23. Given that $\dfrac{7x - x^2}{(2-x)(x^2+1)} \equiv \dfrac{A}{2-x} + \dfrac{Bx + C}{x^2 + 1}$, determine the values of *A*, *B* and *C*. A curve has equation $y = \dfrac{7x - x^2}{(2-x)(x^2+1)}$. Prove that the area of the region enclosed by the curve, the *x*-axis and the line $x = 1$ is $\frac{7}{2}\ln 2 - \frac{\pi}{4}$. (AEB)

24. Use the substitution $x = 1 + 2\tan\theta$ to evaluate the integral $\int_1^3 \dfrac{x}{x^2 - 2x + 5}\,dx$ giving your answer correct to two decimal places. (AEB)

25. By expressing $2\cos 3\theta\sin\frac{\theta}{2}$ and other similar expressions as the difference of two sines, prove the identity
   $(2\cos 3\theta + 2\cos 2\theta + 2\cos\theta + 1)\sin\frac{\theta}{2} \equiv \sin\frac{7\theta}{2}$.
   Express $\cos 3\theta$ and $\cos 2\theta$ in terms of $\cos\theta$ and deduce the identity
   $(8\cos^3\theta + 4\cos^2\theta - 4\cos\theta - 1)\sin\frac{\theta}{2} \equiv \sin\frac{7\theta}{2}$.
   Hence, or otherwise, show that $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}$ and $\cos\frac{6\pi}{7}$ are the roots of the equation
   $8x^3 + 4x^2 - 4x - 1 = 0$. (AEB)

26. Assuming the identities
   $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$ and
   $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$
   prove that
   $\cos 5\theta \equiv 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta$.

(a) Find the set of values of θ in the interval $0<\theta<\pi$ for which $\cos 5\theta > 16\cos^5\theta$.

(b) Find the general solution, in radians, of the equation $\cos x + 3\cos 3x + \cos 5x = 0$. (AEB)

27. Express $f(\theta) = 4\cos\theta + 3\sin\theta$ in the form $R\cos(\theta-\alpha)$ where $R>0$ and $0<\alpha<\frac{\pi}{2}$.

(a) A rectangle *OABC* is formed from the origin, the point $A(4\cos\theta, 0)$, the point *B*, and the point $C(0, 3\sin\theta)$. State the coordinates of *B* and express the perimeter of the rectangle in terms of $f(\theta)$. Hence find the greatest perimeter of the rectangle as θ varies in the range $0 \le \theta \le \frac{\pi}{2}$ and state the coordinates of *B* for which this greatest perimeter occurs.

(b) A curve has the equation $y = \frac{1}{(4\cos x + 3\sin x)^2}\left(0 \le x \le \frac{\pi}{2}\right)$. Show that the region enclosed by the curve, the *x*-axis, and the lines $x=0$ and $x=\frac{\pi}{2}$ has area $\frac{1}{12}$.

(AEB)