

5 VECTOR GEOMETRY

Objectives

After studying this chapter you should

- be able to find and use the vector equation of a straight line;
- be able to find the equation of a plane in various forms;
- be able to interchange between cartesian and vector equations;
- be able to find the perpendicular distance of a point from a plane;
- be able to find the angle between two planes.

5.0 Introduction

This chapter deals with the use of vectors in geometric problems. The key to confidence in answering problems is to be able to visualise the situations. It is usually very helpful to illustrate the situation with a drawing, even though drawing planes can be quite difficult. First, though, try the activity below, which should help you to think geometrically.

Activity 1

Consider the following statements, decide whether each is sometimes true, always true or false, and discuss your answers with your tutor. Think carefully before giving your answers.

- (a) A line and a plane intersect at a point.
 - (b) Two lines intersect at a point.
 - (c) Two planes intersect in a line.
 - (d) A line is uniquely defined by two distinct points on it.
 - (e) A plane is uniquely defined by three distinct points on it.
 - (f) A plane is defined by giving the direction perpendicular to the plane and a point on the plane.
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5.1 Straight line

A straight line, L , is uniquely defined by giving two distinct points on the line.

Are there other ways to define uniquely a straight line?

If the coordinates of A and B are given, then the vectors $\mathbf{a} = \vec{OA}$ and $\mathbf{b} = \vec{OB}$ are known. Let P be any point on the line AB with position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Then
$$\mathbf{r} = \vec{OP} = \vec{OA} + \vec{AP}$$

$$\Rightarrow \mathbf{r} = \mathbf{a} + \vec{AP}$$

But \vec{AP} is a linear multiple of $\vec{AB} = \mathbf{b} - \mathbf{a}$;

so
$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

for some parameter λ . This is the form of the vector equation of a line. The parameter can take any real value, giving different points on the line.

What value of λ gives the point A ?

What value of λ gives the point B ?

Example

Find the vector equation of the straight line passing through the points $A(1, 0, 1)$ and $B(0, 1, 3)$.

Solution

Here $\mathbf{a} = \mathbf{i} + \mathbf{k}$

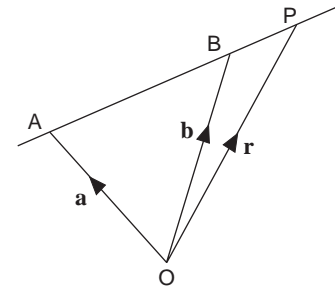
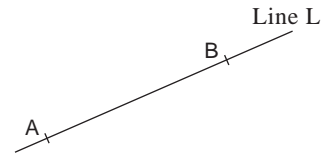
$$\mathbf{b} = \mathbf{j} + 3\mathbf{k}$$

so
$$\mathbf{r} = (\mathbf{i} + \mathbf{k}) + \lambda((\mathbf{j} + 3\mathbf{k}) - (\mathbf{i} + \mathbf{k}))$$

$$\mathbf{r} = (\mathbf{i} + \mathbf{k}) + \lambda(-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \quad (\text{or } (1, 0, 1) + \lambda(-1, 1, 2))$$

The vector equation of a line can readily be turned into a cartesian equation by noting that the coordinates of the point on the line are

$$(x, y, z) = (1 - \lambda, \lambda, 1 + 2\lambda)$$



This gives

$$\left. \begin{array}{l} y = 1 - \lambda \\ y = \lambda \\ z = 1 + 2\lambda \end{array} \right\} \text{ or } \frac{x-1}{(-1)} = \frac{y-0}{1} = \frac{z-1}{2} \quad (= \lambda)$$

This is the form of the cartesian equation of a straight line. In general, the vector equation can be written as

$$\boxed{\mathbf{r} = \mathbf{a} + \lambda \mathbf{t}}$$

where \mathbf{t} is a vector in the direction of the line.

If $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ and $\mathbf{t} = t_1 \mathbf{i} + t_2 \mathbf{j} + t_3 \mathbf{k}$

then

$$\left. \begin{array}{l} x = a_1 + \lambda t_1 \\ y = a_2 + \lambda t_2 \\ z = a_3 + \lambda t_3 \end{array} \right\} \text{ or } \frac{x-a_1}{t_1} = \frac{y-a_2}{t_2} = \frac{z-a_3}{t_3}$$

Example

The cartesian equation of a straight line is given by

$$\frac{x-1}{2} = \frac{y+1}{4} = \frac{z-2}{(-2)}$$

Rewrite it in vector form.

Solution

Writing $\frac{x-1}{2} = \frac{y+1}{4} = \frac{z-2}{(-2)} = \lambda$ for some parameter λ , then

$$\left. \begin{array}{l} x = 1 + 2\lambda \\ y = -1 + 4\lambda \\ z = 2 - 2\lambda \end{array} \right\} \text{ or } \mathbf{r} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \lambda(2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

Note that this equation is not unique (although the line is unique).

For example, you can write

$$\mathbf{r} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + 2\lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

and with $\mu = 2\lambda$,

$$\mathbf{r} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

Similarly, check for yourself that

$$r = (-3j + 3k) + \sigma(i + 2j - k)$$

and

$$r = (3i + 3j) + \rho(-i - 2j + k)$$

describe the same line. In all cases the first vector is on the line and the second is parallel to the line.

Example

Find a vector equation of the line which passes through the point

A (1, -1, 0) and is parallel to the line \vec{BC} where B and C are the points with coordinates (-3, 2, 1) and (2, 1, 0). Show that the point D (-14, 2, 3) lies on the line.

Solution

The line required is parallel to the line \vec{BC} , which has equation

$$(2 - (-3))i + (1 - 2)j + (0 - 1)k = 5i - j - k.$$

Its equation is given by

$$r = i - j + \lambda(5i - j - k).$$

To show that the point D lies on the line, you must check whether

$$r = i - j + \lambda(5i - j - k)$$

can ever equal $-14i + 2j + 3k$ for some value of λ .

So you need

$$[\hat{i}] 1 + 5\lambda = -14 \quad [\hat{j}] 2 = -1 - \lambda \quad [\hat{k}] 3 = -\lambda$$

and all three of these **are** satisfied when $\lambda = -3$. Hence D does lie on the line.

Intersection of two lines

Two non-parallel lines either do not intersect or intersect at a point. Lines which do not intersect are called **skew lines**.

Example

The lines L and M have vector equations

$$L: \mathbf{r} = 2\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j})$$

$$M: \mathbf{r} = \mathbf{i} + \mathbf{j} - 2\mathbf{k} + \mu(\mathbf{j} - \mathbf{k}) \quad (\lambda, \mu \text{ parameters})$$

Show that these two lines intersect and find their point of intersection.

Solution

If the lines intersect, then for some value of λ and μ ,

$$\mathbf{r} = \lambda \mathbf{i} + (2 - \lambda)\mathbf{j} - 2\mathbf{k} = \mathbf{i} + (1 + \mu)\mathbf{j} - (2 + \mu)\mathbf{k}.$$

Equating coefficients of \mathbf{i} , \mathbf{j} , \mathbf{k} gives

$$[\mathbf{i}] \quad \lambda = 1$$

$$[\mathbf{j}] \quad 2 - \lambda = 1 + \mu$$

$$[\mathbf{k}] \quad -2 = -(2 + \mu)$$

The first equation gives $\lambda = 1$ and the second then gives $\mu = 0$. These values of λ and μ also satisfy the third equation and so the lines intersect. To find the point of intersection, put $\lambda = 1$ in the equation for the line L. This gives

$$\mathbf{r} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

[You can check this answer by substituting $\mu = 0$ in the equation for M. This gives $\mathbf{r} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ as before.]

Activity 2

The three lines, L, M and N, have vector equations

$$L: \mathbf{r} = (1 + t)\mathbf{i} + (1 - t)\mathbf{j} - 2\mathbf{k}$$

$$M: \mathbf{r} = (3 - \mu)\mathbf{i} - (1 - \mu)\mathbf{j} - (\mu - 2)\mathbf{k}$$

$$N: \mathbf{r} = (1 + s)\mathbf{i} - (1 + 3s)\mathbf{j} - s\mathbf{k}$$

for parameters t , μ and s . Which pairs of lines intersect?

Exercise 5A

- Find the vector and cartesian equations of the straight line joining the points A and B, whose coordinates are $(-2, 1, 4)$ and $(1, 7, 6)$ respectively.
- If $\vec{OA} = i - 2j + k$, $\vec{OB} = -i - 3k$,
 $\vec{OC} = 3i + j - 2k$, $\vec{OD} = 8i + j + 4k$
 find the vector equation of a straight line
 - through A and B;
 - through D parallel to BC;
 - through C parallel to AB.
- The three lines, L, M and N, are given by the equations

$$\text{L: } r = 7i - 3j + 3k + \lambda(3i - 2j + k)$$

$$\text{M: } r = 7i - 2j + 4k + \mu(-2i + j - k)$$

$$\text{N: } r = i + v(j - k)$$
 - Show that L and M intersect and find their point of intersection.
 - Show that L and N do not intersect.
 - Do M and N intersect?
- Prove that a necessary and sufficient condition for three points to be collinear is that their position vectors satisfy the equation

$$\lambda a + \mu b + \nu c = 0$$
 where λ , μ and ν are scalars such that $\lambda + \mu + \nu = 0$.

5.2 Equation of a plane

There are a number of ways of specifying a plane – you can deduce its equation in each case. Here you will consider two specific ways:

- a point on the plane and a perpendicular vector to the plane are given;
- three non-collinear points on the plane are given.

Point and perpendicular vector

Let the given point on the plane be A with $\vec{OA} = a$. Also let n be a vector perpendicular to the plane, i.e. a **normal** to the plane.

If P is any point on the plane, with position vector r , then

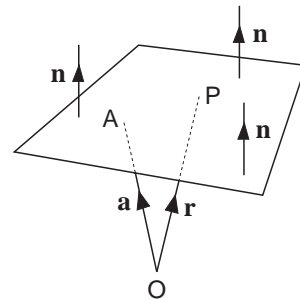
$$\vec{AP} = r - a$$

is perpendicular to n ; so

$$(r - a) \cdot n = 0.$$

Thus the vector equation of a plane is of the form

$$r \cdot n = a \cdot n$$



Example

Find the vector equation of a plane which passes through the point $(0, 1, 1)$ and has normal vector $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Also find its cartesian equation and show that the points $(1, 0, 1)$ and $(1, 1, 0)$ lie on the plane. Sketch the plane.

Solution

The equation is given by

$$\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = (0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\Rightarrow \mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0 + 1 + 1 = 2$$

$$\Rightarrow \mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2.$$

The cartesian equation is found by writing \mathbf{r} as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\text{giving } (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$$

$$\Rightarrow x + y + z = 2.$$

For the point $(1, 0, 1)$, $x = 1$, $y = 0$ and $z = 1$, which satisfies the equation. Similarly for $(1, 1, 0)$.

A sketch of the plane is shown opposite.

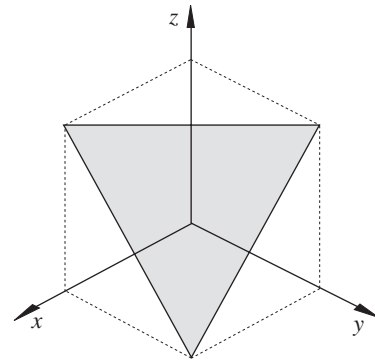
Note that for any particular choice of the normal \mathbf{n} , the equation

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

gives a unique equation for the plane despite the fact that \mathbf{a} is the position vector of any point on the plane. For example, another point on the plane has coordinates $(3, 1, -2)$. In this case, the equation is

$$\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = (3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$$

as before. Different choices for the normal \mathbf{n} (which must be a scalar multiple of $\mathbf{i} + \mathbf{j} + \mathbf{k}$) will give essentially the same equation.



Three non-collinear points

Three non-collinear points are sufficient to define uniquely a plane.

What shape will be defined by four non-collinear points?

Suppose the points A, B and C all lie on the plane and

$$\vec{OA} = \mathbf{a}, \quad \vec{OB} = \mathbf{b}, \quad \vec{OC} = \mathbf{c}.$$

Now the vector $\vec{AB} = \mathbf{b} - \mathbf{a}$ lies in the plane. Similarly

$\vec{AC} = \mathbf{c} - \mathbf{a}$ lies in the plane. Now if P is any point in the plane with position vector \mathbf{r} , then

$$\vec{OP} = \vec{OA} + \vec{AP}$$

$$\mathbf{r} = \mathbf{a} + \vec{AP}$$

By construction you can see that

$$\vec{AP} = \vec{AD} + \vec{DP}$$

where D is on AC, produced such that DP is parallel to AB.

So, if

$$AD = nAC \quad \text{and} \quad DP = mAB$$

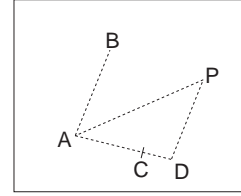
for some parameters m and n , then

$$\begin{aligned} \vec{AP} &= m\vec{AB} + n\vec{AC} \\ &= m(\mathbf{b} - \mathbf{a}) + n(\mathbf{c} - \mathbf{a}) \end{aligned}$$

Finally, you can write the equation as

$$\begin{aligned} \mathbf{r} &= \mathbf{a} + m(\mathbf{b} - \mathbf{a}) + n(\mathbf{c} - \mathbf{a}) \\ &\text{or} \\ \mathbf{r} &= (1 - m - n)\mathbf{a} + m\mathbf{b} + n\mathbf{c} \end{aligned}$$

where m and n are parameters.



Example

Find the vector equation of the plane that passes through the points $(0, 1, 1)$, $(1, 1, 0)$ and $(1, 0, 1)$. Deduce its cartesian form.

Solution

With $\mathbf{a} = \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$, $\mathbf{c} = \mathbf{i} + \mathbf{k}$,

$$\mathbf{r} = \mathbf{j} + \mathbf{k} + m(\mathbf{i} - \mathbf{k}) + n(\mathbf{i} - \mathbf{j})$$

$$\mathbf{r} = (m + n)\mathbf{i} + (1 - n)\mathbf{j} + (1 - m)\mathbf{k}.$$

To find the cartesian equation of the plane, note that

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

so $x = m + n$, $y = 1 - n$, $z = 1 - m$.

Eliminating m and n ,

$$x = (1-z) + (1-y)$$

$$\Rightarrow x + y + z = 2$$

as deduced in the previous example.

Activity 3

Deduce the general equation of a plane passing through the point A, where $\vec{OA} = \mathbf{a}$, and such that the vectors \mathbf{s} and \mathbf{t} are parallel to the plane.

Example

The lines L_1 and L_2 have equations

$$\mathbf{r} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) + \alpha(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

and $\mathbf{r} = (2\mathbf{i} + 5\mathbf{j}) + \beta(\mathbf{i} - \mathbf{j} + \mathbf{k})$

respectively.

- (a) Prove that L_1 and L_2 intersect and find the point of intersection.
- (b) Determine the equation of the plane π containing L_1 and L_2 , giving your answer in the form $\mathbf{r} \cdot \mathbf{n} = d$.

(AEB)

Solution

- (a) If L_1 and L_2 intersect, then

$$(3\mathbf{i} + \mathbf{j} - \mathbf{k}) + \alpha(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} + 5\mathbf{j}) + \beta(\mathbf{i} - \mathbf{j} + \mathbf{k}).$$

Equating coefficients,

$$\left. \begin{array}{l} [\mathbf{i}] \quad 3 + \alpha = 2 + \beta \\ [\mathbf{j}] \quad 1 + 2\alpha = 5 - \beta \end{array} \right\} \begin{array}{l} \alpha - \beta = -1 \\ 2\alpha + \beta = 4 \end{array} \left. \vphantom{\begin{array}{l} [\mathbf{i}] \\ [\mathbf{j}] \end{array}} \right\} 3\alpha = 3 \Rightarrow \alpha = 1, \beta = 2$$

$$[\mathbf{k}] \quad -1 + 3\alpha = \beta$$

Now $\alpha = 1$, $\beta = 2$ satisfies the third equation, and so the lines intersect at the point $(4, 3, 2)$.

(b) The plane contains the point with position vector

$$4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

and the directions of $(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and $(\mathbf{i} - \mathbf{j} + \mathbf{k})$.

Hence

$$\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} + m(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + n(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

for parameters m and n .

[Note that you can find the cartesian form by writing

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \text{ giving}$$

$$\left. \begin{array}{l} x = 4 + m + n \\ y = 3 + 2m - n \\ z = 2 + 3m + n \end{array} \right\} \left. \begin{array}{l} x + y = 7 + 3m \\ y + z = 5 + 5m \end{array} \right\} \Rightarrow 5x + 2y - 3z = 20$$

Activity 4

Show that the equation of a plane, containing the points A, B and C

where $\vec{OA} = \mathbf{a}$, $\vec{OB} = \mathbf{b}$, $\vec{OC} = \mathbf{c}$, can be written in the form

$$\mathbf{r} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c}$$

where $\lambda + \mu + \nu = 1$.

Exercise 5B

- Find the equation of the plane that contains the points A (2, -1, 0), B (-1, 3, 4) and C (3, 0, 2).
- Find the equation of the plane that contains the point A (2, -1, 0) and for which the vector $\mathbf{r} = 4\mathbf{i} + 10\mathbf{j} - 7\mathbf{k}$ is perpendicular to the plane.
- Compare your answers to Questions 1 and 2. What do you deduce?
- Find the equation of the plane passing through (0, 0, 0) and with normal vector $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.
- Show that the line with equation $\mathbf{r} = \mathbf{k} + \lambda(2\mathbf{i} + 6\mathbf{j} + 8\mathbf{k})$ is perpendicular to the plane with equation $x + 3y + 4z = 8$
- Show that the point A (2, 3, 1) lies on the plane $\mathbf{r} \cdot (2\mathbf{i} - \mathbf{k}) = 3$
Also show that the line with vector equation $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ is contained in the plane.
- Find the point of intersection of the line $\mathbf{r} = \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + \mathbf{k})$ with the plane $\mathbf{r} \cdot \mathbf{i} = 4$.
- The planes $\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 19$ and $\mathbf{r} \cdot (-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = -9$ intersect in a line L. Find the cartesian equation of the plane which contains L and is parallel to the vector \mathbf{i} .

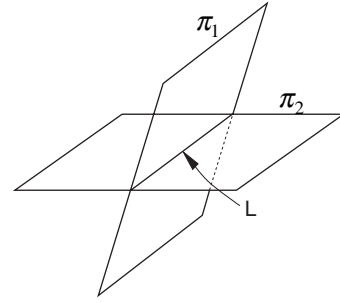
5.3 Miscellaneous problems

Line of intersection of planes

In general, two planes, π_1 and π_2 , intersect in a line, L , as shown opposite.

What else can happen?

The following example illustrates how you can determine the equation of L .



Example

Show that the two planes

$$r \cdot (i + j + k) = 2$$

and $r \cdot (i + 2j + 3k) = 3$

intersect in a line. Find the vector equation of this line.

Solution

The cartesian form of these equations is

$$x + y + z = 2$$

and

$$x + 2y + 3z = 3.$$

Writing $x = \lambda$, (alternatively, you could use $y = \mu$, etc.)

$$y + z = 2 - \lambda \Rightarrow 2y + 2z = 4 - 2\lambda$$

$$2y + 3z = 3 - \lambda.$$

Hence

$$z = 3 - \lambda - (4 - 2\lambda) = -1 + \lambda$$

and $y = 2 - \lambda - z = 2 - \lambda - (-1 + \lambda) = 3 - 2\lambda.$

So the cartesian equation of the line is

$$\frac{x}{1} = \frac{y-3}{(-2)} = \frac{z+1}{1} \quad (= \lambda)$$

or, in vector form

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{t}$$

where $\mathbf{a} = 3\mathbf{j} - \mathbf{k}$

$$\mathbf{t} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

Distance of a point from a plane

To find the distance of the point B with position vector $\mathbf{b} = (b_1, b_2, b_3)$ from the plane with equation

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = d$$

you first find the equation of the line BN, which is perpendicular to the plane. Hence it is in the direction of \mathbf{n} , and so its equation is of the form

$$\mathbf{r} = \mathbf{b} + \lambda \mathbf{n}.$$

But N also lies on the plane, so it satisfies

$$\mathbf{r} \cdot \mathbf{n} = d \Rightarrow (\mathbf{b} + \lambda \mathbf{n}) \cdot \mathbf{n} = d$$

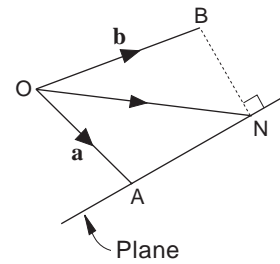
$$\mathbf{b} \cdot \mathbf{n} + \lambda \mathbf{n} \cdot \mathbf{n} = d$$

$$\lambda \mathbf{n} \cdot \mathbf{n} = d - \mathbf{b} \cdot \mathbf{n}$$

$$\lambda = \frac{d - \mathbf{b} \cdot \mathbf{n}}{|\mathbf{n}|^2}$$

But the length of BN is $|\lambda \mathbf{n}| = \frac{|d - \mathbf{b} \cdot \mathbf{n}|}{|\mathbf{n}|}$, so

$$\boxed{\text{distance} = \frac{|d - \mathbf{b} \cdot \mathbf{n}|}{|\mathbf{n}|}}$$



Example

Find the distance of the point $(4, 2, 3)$ from the plane

$$\mathbf{r} \cdot (6\mathbf{i} + 2\mathbf{j} - 9\mathbf{k}) = 46.$$

Solution

Here $d = 46$, $\mathbf{n} = 6\mathbf{i} + 2\mathbf{j} - 9\mathbf{k}$, $|\mathbf{n}| = \sqrt{36 + 4 + 81} = 11$, and $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

So $\mathbf{b} \cdot \mathbf{n} = 24 + 4 - 27 = 1$

and $\text{distance} = \frac{|46-1|}{11} = \frac{45}{11}$

This section is completed by looking at a typical exam-type question.

Example

The points A, B and C have position vectors

$$\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \quad \mathbf{i} + 5\mathbf{j} \quad \text{and} \quad 5\mathbf{i} + 6\mathbf{j} - \mathbf{k}$$

respectively, relative to an origin O.

- Show that AB is perpendicular to BC and find the area of the triangle ABC.
- Find the vector product $\vec{\mathbf{AB}} \times \vec{\mathbf{BC}}$. Hence find an equation of the plane ABC in the form $\mathbf{r} \cdot \mathbf{n} = p$.
- The point D has position vector $4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Find the distance of the point D from the plane ABC. Hence show that the volume of the tetrahedron ABCD is equal to 21.
- Give, in cartesian form, the equation of the plane π which contains D and which has the property that for each point E in π the volume of the tetrahedron ABCE is still 21.

(AEB)

Solution

$$\begin{aligned} \text{(a)} \quad \vec{\mathbf{AB}} &= \mathbf{b} - \mathbf{a} \\ &= (\mathbf{i} + 5\mathbf{j}) - (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \\ &= 3\mathbf{j} + 3\mathbf{k} \end{aligned}$$

$$\begin{aligned} \vec{\mathbf{BC}} &= \mathbf{c} - \mathbf{b} \\ &= (5\mathbf{i} + 6\mathbf{j} - \mathbf{k}) - (\mathbf{i} + 5\mathbf{j}) \\ &= 4\mathbf{i} + \mathbf{j} - \mathbf{k} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \vec{\mathbf{AB}} \cdot \vec{\mathbf{BC}} &= (3\mathbf{j} + 3\mathbf{k}) \cdot (4\mathbf{i} + \mathbf{j} - \mathbf{k}) \\ &= 0 \times 4 + 3 \times 1 + 3 \times (-1) \\ &= 0 \end{aligned}$$

Hence AB and BC are perpendicular.

Therefore triangle ABC has a right angle at B and area of

$$\begin{aligned} \text{triangle ABC} &= \frac{1}{2} |\vec{AB}| |\vec{BC}| \\ &= \frac{1}{2} (\sqrt{9+9}) (\sqrt{16+1+1}) \\ &= \frac{1}{2} \sqrt{18} \sqrt{18} \\ &= 9 \end{aligned}$$

$$\begin{aligned} \text{(b) } \vec{AB} \times \vec{BC} &= (3\mathbf{j} + 3\mathbf{k}) \times (4\mathbf{i} + \mathbf{j} - \mathbf{k}) \\ &= (3(-1) - 3(1))\mathbf{i} + (3(4) - 0(-1))\mathbf{j} + (0(1) - 3(4))\mathbf{k} \\ &= -6\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}. \end{aligned}$$

This is perpendicular to the plane ABC and so the equation of the plane takes the form

$$\begin{aligned} r. (-6\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}) &= a. (-6\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}) \\ &= (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (-6\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}) \\ &= -6 + 24 + 36 \\ &= 54 \end{aligned}$$

[Note that you could have used **b** or **c** instead of **a**. You would still obtain 54 on the right-hand-side.]

Thus the equation of the plane is

$$r. (-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 9 \quad (\text{dividing by } 6)$$

(You could have saved a little effort by noting that $-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, being a multiple of $-6\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}$, is perpendicular to the plane and used that for your **n**.)

(c) Using the formula for the distance of $\mathbf{d} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ from the plane $r. (-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 9$ gives

$$\begin{aligned} \text{distance} &= \frac{|9 - (4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})|}{\sqrt{1+4+4}} \\ &= \frac{21}{\sqrt{1+4+4}} \\ &= 7 \end{aligned}$$

$$\begin{aligned}
 \text{Volume of tetrahedron } ABCD &= \frac{1}{3} \times \text{base area} \times \text{height} \\
 &= \frac{1}{3} \times 9 \times 7 \\
 &= 21
 \end{aligned}$$

- (d) The required plane is parallel to the plane ABC, and so, using the equation of the plane found in (b), has equation

$$r \cdot (-i + 2j - 2k) = \text{constant}, K$$

Since it contains D,

$$(4i - j + 3k) \cdot (-i + 2j - 2k) = K$$

$$\Rightarrow -4 - 2 - 6 = K$$

$$\Rightarrow K = -12$$

and the equation is

$$r \cdot (-i + 2j - 2k) = -12.$$

In cartesian form, $r = xi + yj + zk$, gives

$$-x + 2y - 2z = -12$$

(or $x - 2y + 2z = 12$).

Use of vector product

The vector product can be useful when we need a vector perpendicular to two given vectors. For instance, when we are finding the line of intersection of two planes with equations

$$r \cdot n_1 = d_1 \quad \text{and} \quad r \cdot n_2 = d_2$$

the direction of the line of intersection will be $n_1 \times n_2$.

Can you explain why?

Example

Find a vector equation for the line of intersection of the planes with equations

$$r \cdot (i + 2j + 3k) = 6 \quad \text{and} \quad r \cdot (3i - 2k) = 1.$$

Solution

The direction of the line is given by the vector product

$$(i+2j+3k) \times (3i-2k)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \\ -6 \end{bmatrix} \quad \text{(writing the vectors as } 3 \times 1 \text{ column matrices)}$$

The line of intersection is therefore of the form

$$r = a + \lambda(-4i + 11j - 6k)$$

where **a** is the position vector of any point on both of the planes and hence on the line of intersection.

For instance,

$$(1, 1, 1), \left(\frac{1}{3}, \frac{17}{6}, 0\right), \left(0, \frac{15}{4}, -\frac{1}{2}\right)$$

all lie on each of the planes.

A possible equation for the line of intersection is therefore

$$r = (1 + j + k) + \lambda(-4i + 11j - 6k)$$

Angle between two planes

The two planes

$$r \cdot n_1 = d_1 \quad \text{and} \quad r \cdot n_2 = d_2$$

have normals n_1 and n_2 respectively.

The angle between the two planes is equal to the angle between n_1 and n_2 .

Can you draw a diagram to explain why?

Example

Find the cosine of the acute angle between the two planes

$$r \cdot (i - j + 5k) = 2 \quad \text{and} \quad r \cdot (3i + 2j - k) = 5.$$

Solution

Let $\mathbf{n}_1 = i - j + 5k$ and $\mathbf{n}_2 = 3i + 2j - k$

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 3 - 2 - 5 = -4$$

$$|\mathbf{n}_1| = \sqrt{(1+1+25)} = \sqrt{27}; \quad |\mathbf{n}_2| = \sqrt{(9+4+1)} = \sqrt{14}$$

If θ is the angle between \mathbf{n}_1 and \mathbf{n}_2 ,

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \frac{-4}{\sqrt{27} \times \sqrt{14}} = \frac{-4}{\sqrt{378}}$$

When two lines intersect, the angle between the lines could be taken as the acute or obtuse angle. The answer above gives the obtuse angle since cosine is negative.

The acute angle is given by

$$\cos^{-1}\left(\frac{4}{\sqrt{378}}\right)$$

But this is equal to the angle between the two planes.

The cosine of the acute angle between the planes is

$$\frac{4}{\sqrt{378}}$$

Angle between a line and a plane

Let the acute angle between the line L with equation

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$$

and the plane π with equation $\mathbf{r} \cdot \mathbf{n} = p$ be θ .

Let the acute angle between the direction vector of the line \mathbf{d} and \mathbf{n} , the normal to the plane π , be ϕ .

Can you see why $\theta + \phi = 90^\circ$?

Therefore $\cos \phi = \frac{\mathbf{n} \cdot \mathbf{d}}{|\mathbf{n}| |\mathbf{d}|}$

or $\sin \theta = \frac{\mathbf{n} \cdot \mathbf{d}}{|\mathbf{n}| |\mathbf{d}|}$

The angle between L and π is therefore

$$\sin^{-1}\left(\frac{n \cdot d}{|n||d|}\right)$$

Example

Find the angle between the line with equation

$$r = (2i + k) + \lambda(3i - 4j + k)$$

and the plane with equation

$$r \cdot (5i + j - 6k) = 2$$

Solution

Direction of line $d = 3i - 4j + k$

Normal to plane $n = 5i + j - 6k$

$$d \cdot n = 15 - 4 - 6 = 5 \Rightarrow |d| = \sqrt{26}, \quad |n| = \sqrt{62}$$

Angle between line and normal to plane

$$\begin{aligned} &= \cos^{-1}\left(\frac{5}{\sqrt{26} \sqrt{62}}\right) \\ &\approx 82.8^\circ \end{aligned}$$

\Rightarrow Angle between line and plane

$$\begin{aligned} &= 90^\circ - 82.8^\circ \\ &\approx 7.2^\circ \end{aligned}$$

5.4 Miscellaneous Exercises

- Referred to a fixed origin O , the position vectors of the points A , B , C and D are respectively:

$$-j + k, \quad 2i - j + 3k, \quad -i - 2j + 2k, \quad 7i - 4j + 2k.$$
 - Find a vector which is perpendicular to the plane ABC .
 - Show that the length of the perpendicular from D to the plane ABC is of length $2\sqrt{6}$.
 - Show that the planes ABC and BCD are perpendicular.
 - Find the acute angle between the line BD and the plane ABC , giving your answer to the nearest degree. (AEB)
- The point A has position vector $i + 4j - 3k$ referred to the origin O . The line L has vector equation $r = ti$. The plane π contains the line L and the point A . Find:
 - a vector which is normal to the plane π ;
 - a vector equation for the plane π ;
 - the cosine of the acute angle between OA and the line L .

3. The lines l_1 and l_2 have vector equations

$$l_1: \mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

$$l_2: \mathbf{r} = 4\mathbf{j} + 6\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

- (a) Show that l_1 and l_2 intersect and find the position vector of the point of intersection.
 (b) Find the acute angle between l_1 and l_2 , giving your answer correct to the nearest degree. (AEB)

4. The points P and Q have position vectors

$$\mathbf{p} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{q} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

respectively, relative to a fixed origin O.

- (a) Determine a vector equation of the line l_1 , passing through P and Q in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b}$, where s is a scalar parameter.

- (b) The line l_2 has vector equation

$$\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}).$$

Show that l_1 and l_2 intersect and find the position vector of the point of intersection V.

- (c) Show that PV has length $3\sqrt{11}$.
 (d) The acute angle between l_1 and l_2 is θ . Show that

$$\cos \theta = \frac{3}{\sqrt{11}}$$

- (e) Calculate the perpendicular distance from P to l_2 . (AEB)

5. The position vectors of the points A, B and C are

$$\mathbf{a} = 4\mathbf{i} + 10\mathbf{j} + 6\mathbf{k}, \quad \mathbf{b} = 6\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}, \quad \mathbf{c} = \mathbf{i} + 10\mathbf{j} + 3\mathbf{k}$$

with respect to a fixed origin O.

- (a) Show that the angle ACB is a right angle.
 (b) Find the area of the triangle ABC and hence, or otherwise, show that the shortest distance from C to AB is $3\sqrt{\left(\frac{3}{2}\right)}$.

- (c) The point D lies on the straight line through A and C. Show that the vector $\vec{AD} = \lambda(\mathbf{i} + \mathbf{k})$ for some scalar λ .

Given that the lengths AB and AD are equal, determine the possible position vectors of D. (AEB)

6. Referred to a fixed origin O, the points A, B and C have position vectors

$$3\mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad 7\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} \quad \text{and} \quad \mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

respectively. The vector \mathbf{n} is the vector product

$$\vec{AB} \times \vec{AC}.$$

Express \mathbf{n} in terms of \mathbf{i} , \mathbf{j} and \mathbf{k} and describe the direction of \mathbf{n} in relation to the plane ABC.

Find an equation for the plane ABC in the form $\mathbf{r} \cdot \mathbf{n} = p$. Hence find the shortest distance from O to the plane ABC.

Show that the plane OCA has equation

$$\mathbf{r} \cdot (5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}) = 0.$$

Hence find, to 0.1° , the angle between the plane OCA and the plane ABC. (AEB)

7. The line l has vector equation $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$, where $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{d} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. The points P and Q have position vectors $\mathbf{p} = 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{q} = 2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ respectively. The plane π contains the line l and the point P.

- (a) Find the vector product $\mathbf{d} \times (\mathbf{a} - \mathbf{p})$. Hence, or otherwise, find the equation of the plane π in the form $\mathbf{r} \cdot \mathbf{n} = k$.

- (b) Determine the angle between the lines passing through the points P and Q and the plane π .

- (c) Prove that the point P is equidistant from the line l and the point Q. (AEB)

8. Referred to a fixed origin O, the points

$$A(4, 1, 3), \quad B(-2, 7, 6) \quad \text{and} \quad C(1, 1, 4)$$

have position vectors \mathbf{a} , \mathbf{b} and \mathbf{c} respectively.

Find $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ and hence, or otherwise, determine in the form $\mathbf{r} \cdot \mathbf{n} = p$ an equation of the plane ABC.

The point D with position vector $\mathbf{d} = 5\mathbf{i} - 3\mathbf{j} + \lambda \mathbf{k}$ lies in the plane ABC. Find the value of λ .

Prove that ABCD is a trapezium and find its area. (AEB)

9. With respect to a fixed origin O, the lines l_1 and l_2 are given by the vector equations

$$l_1: \mathbf{r} = 9\mathbf{i} - 4\mathbf{j} + 5\mathbf{k} + t(2\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$l_2: \mathbf{r} = 2\mathbf{i} - 8\mathbf{j} + 12\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}),$$

where t and s are scalar parameters. The point A lies on l_1 and OA is perpendicular to l_1 .

Determine the position vector of A and hence find in the form $\mathbf{r} \cdot \mathbf{n} = p$ an equation of the plane π_1 which passes through A and is perpendicular to OA.

Show that l_1 and l_2 intersect and find the position vector of B, their point of intersection.

Find a vector which is perpendicular to both l_1 and l_2 and hence find an equation for the plane π_2 which contains l_1 and l_2 .

Find, to the nearest one tenth of a degree, the acute angle between the planes π_1 and π_2 .

(AEB)

10. With respect to a fixed origin O , the points L and M have position vectors $6\mathbf{i}+3\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}+2\mathbf{j}+\mathbf{k}$ respectively.
- (a) Form the scalar product $\vec{OL} \cdot \vec{OM}$ and hence find the cosine of angle LOM .
- (b) The point N is on the line LM , produced such that angle MON is 90° . Find an equation for the line LM in the form $\mathbf{r}=\mathbf{a}+b\mathbf{t}$ and hence calculate the position vector of N . (AEB)
11. The plane π_1 contains the points $(1, 4, 2)$, $(1, 0, 5)$ and $(0, 8, -1)$. Find its equation in cartesian form.
- The plane π_2 contains the point $(2, 2, 3)$ and has normal vector $(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$. Find its equation in cartesian form.
- The point $(p, 0, q)$ lies in both the planes π_1 and π_2 . Find p and q and express the equation of the line of intersection of the two planes in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$.
- The point $(1, 1, \mu)$ is equidistant from the planes π_1 and π_2 . Find the two possible values of μ . (AEB)
12. The points A, B and C have position vectors $\mathbf{a}=\mathbf{i}+2\mathbf{j}+4\mathbf{k}$, $\mathbf{b}=-2\mathbf{i}+3\mathbf{j}+5\mathbf{k}$, $\mathbf{c}=3\mathbf{i}-\mathbf{j}+2\mathbf{k}$ respectively, with respect to a fixed origin.
- (a) Show that the point $P(1-3\lambda, 2+\lambda, 4+\lambda)$ lies on the straight line through A and B .
- Express PC^2 in terms of λ and show that, as λ varies, the least value of PC^2 is 6. Verify that in this case the line PC is perpendicular to the line AB .
- (b) Find a vector perpendicular to AB and AC and hence, or otherwise, find an equation for the plane ABC in the form $\mathbf{r} \cdot \mathbf{n} = p$.
- (c) Find a cartesian equation of the plane π which contains the line AB and which is perpendicular to the plane ABC .
- (d) Verify that the point D with position vector $2\mathbf{i}-2\mathbf{j}+11\mathbf{k}$ lies in the plane π and is such that DA is perpendicular to AB . Hence, or otherwise, calculate the volume of the tetrahedron $ABCD$. (AEB)
13. The points A, B and C have position vectors $\mathbf{a}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, $\mathbf{b}=3\mathbf{i}+6\mathbf{j}-5\mathbf{k}$, $\mathbf{c}=7\mathbf{i}+20\mathbf{j}-7\mathbf{k}$ respectively, relative to a fixed origin. The point D has position vector \mathbf{d} and is such that $ABCD$ is a parallelogram.
- (a) Find \mathbf{d} in terms of \mathbf{i}, \mathbf{j} and \mathbf{k} .
- (b) Calculate $(\mathbf{b}-\mathbf{a}) \cdot (\mathbf{c}-\mathbf{a})$ and hence determine the size of angle BAC , giving your answer to the nearest 0.1° .
- (c) Calculate the area of the parallelogram $ABCD$.
- (d) Show that a general point P on the diagonal AC is $(1+2\lambda, 2+6\lambda, 2-3\lambda)$. Write down the vector \vec{BP} and hence, or otherwise, determine the position vector of the point on the line AC that is closest to B . (AEB)
14. With respect to a fixed origin, O , the points A and B have position vectors $(2\mathbf{i}+3\mathbf{j}+6\mathbf{k})$ and $(2\mathbf{i}+4\mathbf{j}+4\mathbf{k})$ respectively.
- (a) Calculate $|\vec{OA}|, |\vec{OB}|$ and, by using the scalar product $\vec{OA} \cdot \vec{OB}$, calculate the value of the cosine of angle AOB .
- (b) The point C has position vector $5\mathbf{i}+12\mathbf{j}+6\mathbf{k}$. Show that OC and AB are perpendicular. Show also that the line through O and C intersects the line through A and B , and find the position vector of the point E where they intersect.
- (c) Given that $\vec{AE} = \lambda \vec{EB}$, find the value of λ and explain briefly why λ is negative. (AEB)