

# 7 CIRCULAR MOTION

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## Objectives

After studying this chapter you should

- appreciate that circular motion requires a force to sustain it;
- know and understand that, for motion in a circle with uniform angular velocity, the acceleration and the force causing it are directed towards the centre of the circle;
- use your knowledge to model applications of circular motion.

## 7.0 Introduction

You will have seen and experienced examples of circular motion throughout your everyday life. Theme park rides such as ‘The Wave Swinger’, ‘The Pirate Ship’, ‘Loop-the-Loop’; cornering on a bicycle or in a car; household equipment such as washing machines, tumble and spin driers, salad driers; specialised devices such as the centrifuge in the chemistry laboratory; the governors of a steam engine. These are just a few of the many examples which you will have come across or been a part of.

The purpose of this chapter is to enable you to analyse circular motion, to help you model some of the situations noted above and be able to predict and explain observations which you can make.

The philosophers of the ancient world considered circular motion to be a natural motion. The heavenly bodies, the planets and the stars, moved in circles around the Earth. Once set upon their paths by the Gods, these bodies continued to move in circles without any further intervention. No force was required to sustain their heavenly orbits.

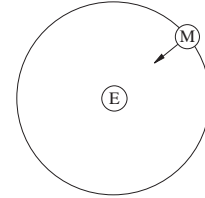
However, from Newton’s point of view, circular motion requires a force to sustain it. Recall Newton’s First Law from Chapter 2,

A body remains in a state of rest or moves with uniform motion, unless acted upon by a force.

A body describing a circle is certainly not at rest. Nor does it move with constant velocity, since, whether the speed is changing or constant, the direction of the motion is changing all the time. From Newton’s First Law, there must be a resultant force acting on the body and Newton’s Second Law equates this force to the mass

times acceleration, so the body must be accelerating.

An example of this is the case of the Moon (M in diagram opposite) orbiting the Earth (E). The Moon describes a path around the Earth which is approximately circular. The force which the Earth exerts upon the Moon is the force of gravity and this pulls the Moon towards the Earth. If this force of gravity did not act, the Moon would move off at a tangent to its path around the Earth. Indeed Newton envisaged the motion of the Moon around the Earth as a series of steps, a tangential movement followed by a move in towards the Earth. If you make the time intervals sufficiently small, this saw-tooth curve becomes a circle.



A further consequence is that since the force upon the Moon is directed towards the Earth so the acceleration is also in this direction.

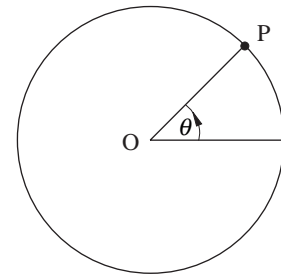
A body moving in a circle must

- (a) from Newton's First Law, have a resultant force acting on it;
- (b) from Newton's Second Law, be accelerating in a specific direction.

## 7.1 Angular velocity and angular speed

As a particle P moves around the circle, centre O, the radius OP turns through an angle  $\theta$ , measured from a fixed radius. The conventions are

- (a) anti-clockwise rotation is positive,
- (b)  $\theta$  is measured in radians.



The rate at which  $\theta$  is changing with respect to time is  $\frac{d\theta}{dt}$ .

This is called the **angular velocity** of the particle P.

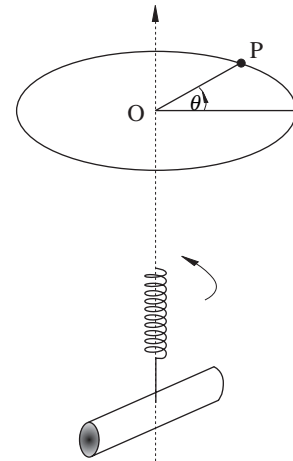
The angular speed is  $\omega$ , the magnitude of  $\frac{d\theta}{dt}$ , and is measured in radians per second.

$$\omega = \left| \frac{d\theta}{dt} \right|.$$

You may find it strange that  $\frac{d\theta}{dt}$  is called the angular velocity.

The word velocity conjures up thoughts of a vector and yet  $\frac{d\theta}{dt}$  is not written as a vector, nor does it seem to have a direction associated with it.

However, think of a corkscrew below the plane of the circle and pointing vertically upwards towards the centre of the circle. As the corkscrew rotates anti-clockwise, i.e. right-handedly, when viewed from above, it will advance along the vertical axis towards and through the centre of the circle. It is this direction of motion of the corkscrew that is the direction associated with the angular velocity  $\frac{d\theta}{dt}$ .



### Activity 1 The relation between velocity and angular velocity

For this activity you will need a piece of polar graph paper, a ruler, a protractor, or a piece of polar graph paper and a programmable calculator.

A piece of polar graph paper can be used to represent the path of the particle moving in a circle with constant angular speed, the radius of the circle sweeping out equal angles in equal intervals of time.

If the angular speed of the particle is  $2\pi \text{ rad s}^{-1}$ , then it completes one revolution every second and each of the points at the ends of the 36 radii drawn represent the positions of the

particle every  $\frac{1}{36}$  of a second. (Remember:  $2\pi \text{ radians} \equiv 360^\circ$ .)

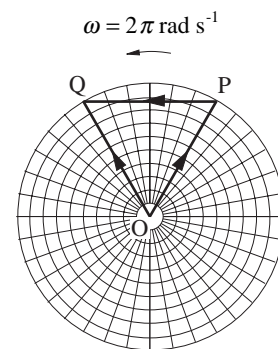
In the diagram opposite, the angle POQ is  $\frac{\pi}{3}$  radians and the

particle has taken  $\frac{6}{36} = \frac{1}{6}$  seconds to travel from P to Q around the circle.

The average velocity of the particle is

$$\frac{\vec{OQ} - \vec{OP}}{t} = \frac{\vec{PQ}}{t}$$

where  $t = \frac{1}{6}$  seconds, the time taken to travel from P to Q.



The magnitude of the average velocity of the particle is

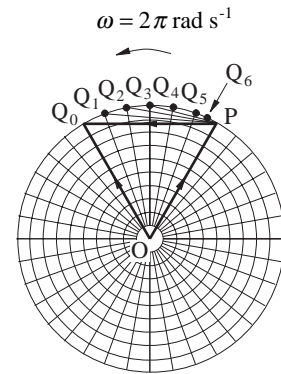
$$\left| \frac{\vec{PQ}}{t} \right| = \frac{PQ}{t}$$

$\frac{PQ}{t}$  is not the average speed of the particle. Why not?

Now investigate the average velocity of the particle for different times  $t$ .

Copy and complete the table below, using the diagram opposite.

angle swept out	time in secs	distance PQ	$\frac{PQ}{t}$ ms <sup>-1</sup>
$\frac{\pi}{3}$	$\frac{1}{6}$	PQ <sub>0</sub> =	
$\frac{2\pi}{18}$	$\frac{5}{36}$	PQ <sub>1</sub> =	
$\frac{2\pi}{9}$			
$\frac{\pi}{6}$			
$\frac{\pi}{9}$			
$\frac{\pi}{18}$			
$\frac{\pi}{36}$			



What value does  $\frac{PQ}{t}$  approach as  $t$  gets smaller?

The average velocity,  $\frac{\vec{PQ}}{t}$ , as  $t$  becomes smaller and smaller approaches the velocity of the particle at P.

What is the magnitude of the velocity at P?

Is there a relation between this magnitude,  $\omega$ , and  $r$ , the radius of the circle in metres?

In which direction is the velocity at P?

## Exercise 7A

- What is the angular speed of the tip of the minute hand of a clock in:
  - revolutions per minute;
  - degrees per second;
  - radians per second?
- A particle has a constant angular speed  $\omega$  and is moving in a circle of radius  $r$ .
  - What is the time for one revolution?
  - How far has the particle travelled in one revolution?
  - What is the average speed of the particle?
- Estimate the angular speed of the Moon as it orbits the Earth.
 

The distance of the Moon from the Earth is approximately 355,000 km. Calculate the speed of the Moon relative to the Earth.
- Estimate the angular speed of the Earth as it orbits the Sun. (Assume that the orbit is a circular path.)
 

The distance of the Earth from the Sun is approximately  $150 \times 10^6$  km. Calculate the speed of the Earth relative to the Sun.

## 7.2 Describing motion in a circle with vectors

Activity 1 suggests that for a particle moving in a circle with constant angular speed  $\omega$ , the velocity of the particle at any instant is of magnitude  $r\omega$  and is directed along the tangent to the circle at the position of the particle. This section begins with a proof of this result.

A particle P, on a circle of radius  $r$ , has coordinates:

$$(r\cos\theta, r\sin\theta) \quad (1)$$

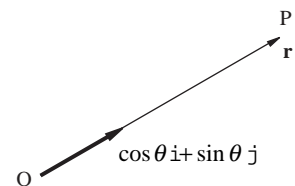
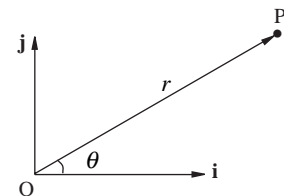
where O is the origin and the centre of the circle;  $\theta$  is the angle, measured in radians, which the radius OP makes with the positive direction of the  $x$ -axis.

As P moves around the circle  $r$  remains constant and  $\theta$  varies with  $t$ , the time. At any instant of time the position vector of P with respect to the origin O referred to the axes shown is,

$$\begin{aligned} \mathbf{r} &= \overrightarrow{OP} = r\cos\theta \mathbf{i} + r\sin\theta \mathbf{j} \\ \Rightarrow \mathbf{r} &= r(\cos\theta \mathbf{i} + \sin\theta \mathbf{j}) \end{aligned} \quad (2)$$

The magnitude of the vector  $\cos\theta \mathbf{i} + \sin\theta \mathbf{j}$  is  $\sqrt{\cos^2\theta + \sin^2\theta} = 1$ , so this vector is the unit vector in the direction of the radius OP and rotates with P as P moves round the circle. The position vector

$$\mathbf{r} = r(\cos\theta \mathbf{i} + \sin\theta \mathbf{j})$$



then has magnitude  $r$  and is in the direction of the unit vector

$(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$ .

The velocity,  $\mathbf{v}$ , of the particle is then given by

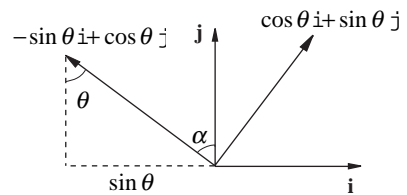
$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})) \\ &= r \left( \frac{d}{dt}(\cos \theta) \mathbf{i} + \frac{d}{dt}(\sin \theta) \mathbf{j} \right) \\ &= r \left( \frac{d}{d\theta}(\cos \theta) \frac{d\theta}{dt} \mathbf{i} + \frac{d}{d\theta}(\sin \theta) \frac{d\theta}{dt} \mathbf{j} \right) \end{aligned}$$

using the 'function of a function' rule for differentiation.

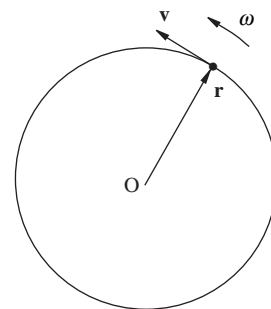
Hence, 
$$\mathbf{v} = r \left( -\sin \theta \frac{d\theta}{dt} \mathbf{i} + \cos \theta \frac{d\theta}{dt} \mathbf{j} \right) \quad (3)$$

Again,  $(-\sin \theta)^2 + (\cos \theta)^2 = 1$ , thus the vector  $(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j})$  is also a unit vector, but what is its direction?

From the diagram opposite, it is clear that  $\alpha = \theta$  so that the vector  $-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$  is perpendicular to the vector  $\cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ . This means that the direction of the velocity  $\mathbf{v}$  is perpendicular to the direction of the radius vector  $\mathbf{r}$ .



Hence the velocity at P is along the tangent to the circle in the direction  $\theta$  increasing when  $\frac{d\theta}{dt}$  is positive and in the opposite direction when  $\frac{d\theta}{dt}$  is negative.



The magnitude of the velocity  $\mathbf{v}$ , is

$$\begin{aligned} |\mathbf{v}| = v &= \left| r \frac{d\theta}{dt} (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \right| \\ &= r \left| \frac{d\theta}{dt} \right| \\ &= r\omega. \end{aligned}$$

The results you should have obtained from Activity 1 are therefore validated.

## Summary

The velocity of a particle P moving in a circle of radius  $r$  with angular speed  $\omega$  is of magnitude  $r\omega$  and is along the tangent to the circle at P in the direction of  $\theta$  increasing when  $\frac{d\theta}{dt}$  is positive and in the opposite direction when  $\frac{d\theta}{dt}$  is negative.

In the introduction to this chapter you saw that any particle moving in a circle must be accelerating. Using the system of unit vectors which has just been set up, this acceleration can now be found.

From equation (3) above,

$$\mathbf{v} = r \frac{d\theta}{dt} (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}).$$

Hence the acceleration,  $\mathbf{a}$ , is given by

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt} \left\{ r \frac{d\theta}{dt} (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \right\} \\ &= r \frac{d\theta}{dt} \frac{d}{dt} (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) + (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \frac{d}{dt} \left( r \frac{d\theta}{dt} \right) \end{aligned}$$

using the product rule for differentiation. (Remember that  $r$  is constant.)

Hence

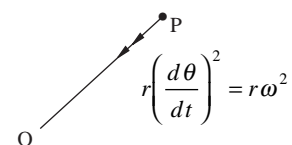
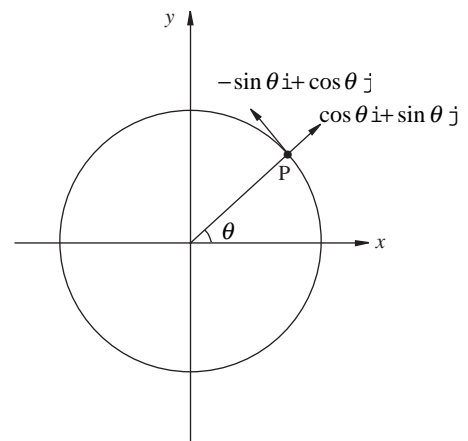
$$\mathbf{a} = r \frac{d\theta}{dt} \left( -\cos \theta \frac{d\theta}{dt} \mathbf{i} - \sin \theta \frac{d\theta}{dt} \mathbf{j} \right) + r \frac{d^2\theta}{dt^2} (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j})$$

using the function of a function rule for differentiation, and so

$$\mathbf{a} = -r \left( \frac{d\theta}{dt} \right)^2 (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) + r \frac{d^2\theta}{dt^2} (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}).$$

This expression is the sum of two vectors. One of these,

$-r \left( \frac{d\theta}{dt} \right)^2 (\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$ , is associated with the unit vector  $(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$  which is in the direction from O to P.



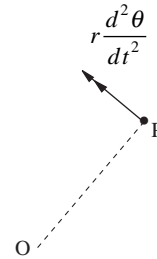
Since  $\left(\frac{d\theta}{dt}\right)^2$  is positive,  $-r\left(\frac{d\theta}{dt}\right)^2$  is negative and hence the vector is in the direction from P to O, that is radially inwards.

Its magnitude is  $r\left(\frac{d\theta}{dt}\right)^2$ , which is equal to  $r\omega^2$ .

Since  $v = r\omega$ , this can be written as  $\frac{v^2}{r}$ .

The second vector,  $r\frac{d^2\theta}{dt^2}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$ , is associated with the unit vector  $(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$ , which is parallel to the tangent at

P. Its magnitude is  $r\left|\frac{d^2\theta}{dt^2}\right|$ .



In the case when  $\frac{d\theta}{dt}$ , and hence the angular speed  $\omega = \left|\frac{d\theta}{dt}\right|$  are

constant,  $\frac{d^2\theta}{dt^2} = 0$ .

## Summary

When a particle is moving in a circle with constant angular speed  $\omega$ , there is only one component of acceleration. This is  $r\omega^2$ , and it is directed towards the centre of the circle.

### Example

A car has wheels which are 75 cm in diameter and is travelling at a constant speed of 60 kph. If there is no slipping between the wheels and the road, calculate the angular speed and the acceleration of a stone wedged in the tread of one of the tyres.

### Solution

Converting the speed to  $\text{ms}^{-1}$  gives

$$\begin{aligned} v &= 60 \text{ kph} = \frac{60 \times 1000}{60 \times 60} \text{ ms}^{-1} \\ &= 16.6 \text{ ms}^{-1} \text{ (to 3 sig. fig.)} \end{aligned}$$



The stone describes circular motion with constant angular speed, so  $v = r\omega$ , and since  $r = 37.5 \text{ cm} = 0.375 \text{ m}$ , then

$$\omega = \frac{v}{r} = 44.3 \text{ rads s}^{-1} \quad (\text{to 3 sig. fig.}).$$

For circular motion with constant angular speed  $\omega$ , the acceleration is  $r\omega^2$  radially inwards; hence

$$a = 736 \text{ ms}^{-2}.$$

*How might your answer be changed if there is slipping between the wheels and the road?*

### Example

A particle is moving in a circle of radius 2 m such that the angle swept out by the radius is given by  $\theta = t^2$ . Calculate the angular velocity, the speed and the acceleration of the particle, giving your answers in terms of  $t$ .

### Solution

Since  $\theta = t^2$ ,

differentiating  $\theta$  gives the angular speed of the particle,  $\omega$ , as

$$\omega = \frac{d\theta}{dt} = 2t \text{ rads s}^{-1}.$$

The actual speed of the particle,  $v$ , is given by

$$v = r\omega.$$

Substituting for  $r = 2$  and  $\omega = 2t$  gives

$$v = 4t \text{ ms}^{-1}.$$

The acceleration has two components since the angular speed is not constant; these components are

$$\text{radial component} = r\omega^2 = 8t^2,$$

$$\text{tangential component} = \frac{rd^2\theta}{dt^2} =, \text{ using } \frac{d^2\theta}{dt^2} = 2.$$

Hence the acceleration is  $8t^2 \text{ ms}^{-2}$  directed radially inwards and  $4 \text{ ms}^{-2}$  tangentially in the direction of increasing  $\theta$ .

## exercise 7B

- A bicycle wheel has a diameter of 90 cm and is made to turn so that a spoke sweeps out an angle  $\theta$  in time  $t$  given by  $\theta = 2t$  radians. The bicycle is stationary. Calculate the angular velocity, the speed and the acceleration of a point on the rim of the wheel.
- The acceleration of the valve of a tyre, which is rotating, is known to be radially inwards and of magnitude  $120 \text{ ms}^{-2}$ .  
If the distance of the valve from the centre of the hub of the wheel is 40 cm, calculate the angular speed of the wheel and the speed of the valve.
- In each of the following cases, the angle  $\theta$  swept out in time  $t$  for a particle moving in a circle of radius 1 m is given as a function of  $t$ . Calculate the components of velocity and acceleration in each case:
  - $\theta = t$
  - $\theta = t^3$ .
- A record turntable has a radius of 15 cm and takes 4 seconds to reach an angular speed of 33 rpm from rest. Assuming that the angular speed,  $\omega$ , increases at a constant rate, express  $\omega$  as a function of  $t$  and calculate the components of acceleration of a point on the rim of the turntable after 2 seconds.

## 7.3 Motion in a circle with constant angular speed

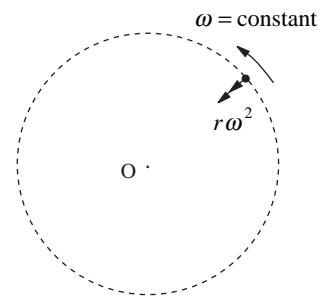
In section 7.2 you saw that the acceleration of a particle P moving in a circle of radius  $r$  metres with constant angular speed

$\omega \text{ rad s}^{-1}$  has magnitude  $r\omega^2$  or  $\frac{v^2}{r}$  and is directed

inwards along the radius from P towards the centre of the circle.

From Newton's Second Law you can conclude that the resultant force on the particle must also act in this direction and has

magnitude  $mr\omega^2$  or  $\frac{mv^2}{r}$ , where  $m$  is the mass of the particle.



### Activity 2 Investigating the forces in circular motion

For this activity you will need a circular cake tin or a plastic bucket and a marble.

Place the marble inside the cake tin.

Holding the cake tin horizontally, move it around so that the marble rolls around the sides of the tin, in contact with the base.

You should be able to get the marble going fast enough so that for a short period of time you can hold the tin still and the marble will describe horizontal circles with uniform angular velocity.

What do you feel through your hands?

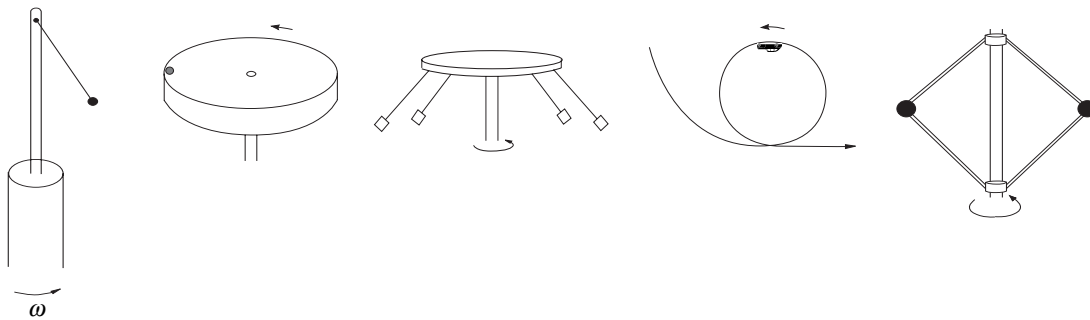
What are the forces acting on the marble?

Draw a force diagram for the forces acting on the marble.

What is the magnitude and direction of the resultant force acting on the marble?

### Activity 3 Force and acceleration

For each of the following situations, mark on the diagram the forces which are acting on the particle, the direction of the resultant force and the acceleration.



### Example

The marble of Activity 2 has mass 10 g and the cake tin a radius of 15 cm. If the marble rolls round the bottom of the tin in contact with the side at an angular speed of  $20 \text{ rad s}^{-1}$ , what are the normal contact forces of the side and base of the tin on the marble?

### Solution

The forces on the marble are:

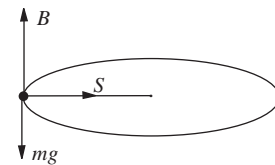
- its weight  $0.01 \text{ g N}$  vertically downwards;
- the normal contact force  $B$  newtons of the base on it;
- the normal contact force  $S$  newtons of the side on it.

The acceleration of the marble is radially inwards and has magnitude

$$\omega^2 r = (20)^2 \times 0.15 = 60 \text{ ms}^{-2}.$$

Applying Newton's Second Law in the radial direction gives

$$S = m\omega^2 r = 0.01 \times 60 = 0.6 \text{ N}.$$



Since there is no acceleration vertically,

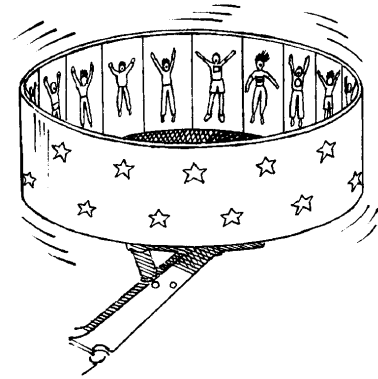
$$B - 0.01g = 0,$$

so  $B = 0.1 \text{ N}$ .

## The rotor ride

What are the forces acting on the people in the ride shown opposite?

Is there a critical angular speed at which this ride must operate to achieve the effects on the people which the diagram shows?



### Modelling the ride

Set up the model by considering each person as a particle, and looking at the forces acting upon it. You looked at a physical situation very similar to this in Activity 2, so you should have some idea of the forces acting.

The forces acting are:-

- the weight,  $mg$ , acting vertically downwards;
- the force of friction,  $F$ , acting vertically upwards and preventing the particle sliding down the wall;
- the normal contact force,  $N$ , exerted radially inwards by the wall on the particle.

The particle is moving in a circle, radius  $r$ , with constant angular speed  $\omega$ , so there is an acceleration  $r\omega^2$  directed along the radius towards the centre of the circle.

Considering the forces in the two mutually perpendicular directions, vertically and radially, and applying Newton's Second Law in each direction you have,

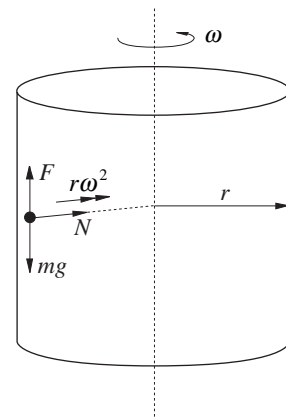
$$\text{vertically} \quad F - mg = 0,$$

$$\text{radially} \quad N = mr\omega^2.$$

But for equilibrium,  $F \leq \mu N$

and therefore  $mg \leq \mu mr\omega^2$

$$\Rightarrow \omega^2 \geq \frac{g}{r\mu}.$$



Therefore there is a critical angular speed,

$$\omega = \sqrt{\frac{g}{r\mu}},$$

which is independent of the mass of the particle or person. This is clearly shown by the people on the ride itself. No matter what their size is, they all behave in exactly the same way. The angular speed,  $\omega$ , must be greater than the critical angular speed above, for the effects to be observed.

For a particular ride, the diameter of the drum is about 10 metres and the coefficient of friction is 0.5. If  $g$  is taken to be  $10 \text{ ms}^{-2}$ , then the critical angular speed

$$\begin{aligned}\omega &= \sqrt{\frac{10}{(5 \times 0.5)}} = 2 \text{ rad s}^{-1} \\ &= 19 \text{ rpm.}\end{aligned}$$

Therefore the ride will have to be rotating in excess of this angular speed for the effect of people 'sticking' to the wall to be observed.

#### Activity 4 The conical pendulum

For this activity you will need either a mass on the end of a piece of string or a mechanics kit containing a conical pendulum.

A bob is made to describe circles which are in a horizontal plane. If you have access to a suitable mechanics kit you can set it up as shown opposite. If not, then you can do much the same thing using a mass on the end of a piece of string. This particular form of motion is often referred to as the 'conical pendulum'.

If the angular speed increases, what happens to the bob?

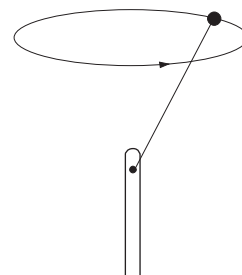
Does the mass of the bob have any effect?

Is it possible to get the bob to describe:

- a circle so that the string is horizontal?
- a circle above the point of suspension rather than below?

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To model this situation, what assumptions would you make?



## Modelling the conical pendulum

### Set up the model

To begin, the following assumptions simplify the problem;

1. the bob is a particle;
2. the string is light and inextensible;
3. the point of suspension is directly above the centre of the horizontal circle described by the bob;
4. the angular speed is constant.

With these assumptions the problem becomes one of motion of a particle in a circle of radius  $r$  about the axis of rotation  $ON$ , with constant angular speed  $\omega$ . The length of the string is  $l$  and it makes an angle  $\theta$  to the axis of rotation.

The forces acting on the particle are the tension  $T$  and the weight  $mg$ .

Since the particle describes circular motion, it has an acceleration  $r\omega^2$  towards the centre of the circle  $N$ .

( $P$  describes a horizontal circle with centre at  $N$ .)

Applying Newton's Second Law vertically and horizontally gives two equations of motion:

$$T \cos \theta - mg = 0 \quad (1)$$

$$T \sin \theta = mr\omega^2. \quad (2)$$

### Solving the equations

The equations are solved to give an expression for  $\theta$ .

From triangle  $ONP$ , the radius  $r$  is given by

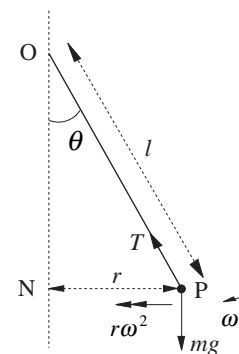
$$r = l \sin \theta.$$

Substituting for  $r$  into equation (2) gives

$$T \sin \theta = ml\omega^2 \sin \theta$$

hence  $T = ml\omega^2$ .

Substituting for  $T$  into equation (1) gives



$$ml\omega^2 \cos \theta = mg$$

hence  $\cos \theta = \frac{g}{l\omega^2}$ . (3)

### Validating the solution

The questions raised in Activity 4, relating to the conical pendulum, can now be explored mathematically using the solution in equation (3).

### Activity 5 The conical pendulum revisited

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What happens to  $\cos \theta$  as  $\omega$  changes?

Does the mass affect the angle  $\theta$ ?

What is the maximum value of  $\theta$ ? Is it realistically possible to achieve this value?

Find the minimum value of the angular speed  $\omega$  for the mass to describe circular motion.

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Do your answers to Activity 3 agree with the mathematical model?

### Activity 6 The chair-o-planes ride

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Look carefully at the picture of the chair-o-planes ride.

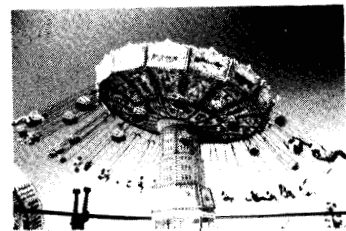
What might happen if the solution depended on mass?

How would you amend the model of the conical pendulum to model this ride?

Find a formula for the angle that each supporting rope makes with the vertical.

It may be possible to visit a fair or a park and, by calculating some distances, estimate the angular speed of the ride.

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## Exercise 7C

- A particle of mass 2 kg is moving in a circle of radius 5 m with a constant speed of  $3 \text{ ms}^{-1}$ . What is the magnitude and direction of the resultant force acting on the particle?
- A penny is placed on the turntable of a record player 0.1 m from the centre. The turntable rotates at 45 rpm. If the penny is on the point of slipping, calculate the coefficient of friction between the penny and the turntable.  
Calculate the resultant force acting on the penny in terms of  $m$ , the mass of the penny, when the turntable is rotating at 33 rpm.
- An inextensible string has a length of 3 m and is fixed at one end to a point O on a smooth horizontal table. A particle of mass 2 kg is attached to the other end and describes circles on the table with O as centre and the string taut. If the string breaks when the tension is 90 N, what is the maximum speed of the particle?
- A particle of mass 4 kg is attached by a light inextensible string of length 3 m to a fixed point. The particle moves in a horizontal circle with an angular speed of  $2 \text{ rads}^{-1}$ .  
Calculate:  
(a) the tension in the string;  
(b) the angle the string makes with the vertical;  
(c) the radius of the circle.
- A particle of mass 5 kg is attached to a fixed point by a string of length 1 m. It describes horizontal circles of radius 0.5 m. Calculate the tension in the string and the speed of the particle.
- A bead of mass  $m$  is threaded on a string of length 8 m which is light and inextensible. The free ends of the string are attached to two fixed points separated vertically by a distance which is half the length of the string, the lower fixed point being on the horizontal table. The bead is made to describe horizontal circles on the table around the lower fixed point, with the string taut. What is the maximum value of  $\omega$ , the angular speed of the bead, if it is to remain in contact with the table?
- A geostationary satellite orbits the Earth in such a way that it appears to remain stationary over a fixed point on the equator. The satellite is actually orbiting the Earth.  
How long does one orbit take?  
What is the radius of the orbit?  
How far away from the surface of the Earth is the satellite?  
What is the speed of the satellite?  
Does the mass of the satellite matter?  
(The Universal Constant of Gravitation,  
 $G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$ .  
Mass of the Earth =  $6 \times 10^{24} \text{ kg}$ .  
Radius of the Earth =  $6 \times 10^3 \text{ km}$ .)

## 7.4 Motion on a banked curve

Why do birds need to bank in order to change the direction of their flight?

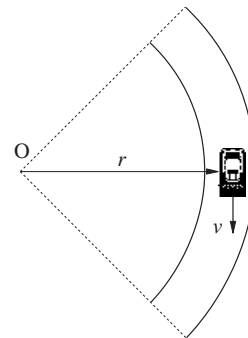
What advantages are gained?

### Activity 7 Going round the bend

The diagram shows a car taking a bend at a constant speed  $v$ .

What is the greatest speed at which it can take the bend on a horizontal road?

Do you think that the car can take the bend at a greater speed if the bend is banked, than if it is not?





## Modelling the motion of a car on a banked road

Assume that the car is a particle, the bend is a part of a circle and the road is inclined at an angle  $\alpha$  to the horizontal.

The forces acting on the car are its weight  $mg$ , vertically downwards, the normal contact force, perpendicular to the road surface, and the friction force  $F$ . However there is a need to stop and think for a moment about the friction force,  $F$ .

In which direction does it act?

In which directions can it act?

Is it possible for the friction force to be zero?

If the car were on the point of sliding **down** the incline of the bend,  $F$  would act in the direction **up** the incline.

If the car were on the point of sliding **up** the incline,  $F$  would act in the direction **down** the incline.

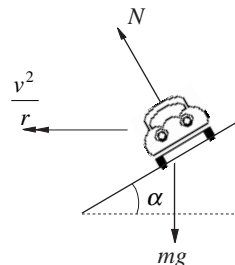
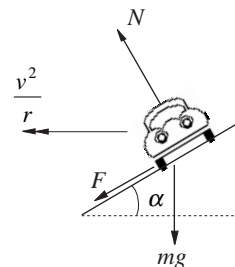
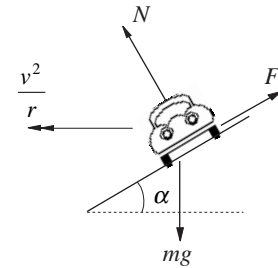
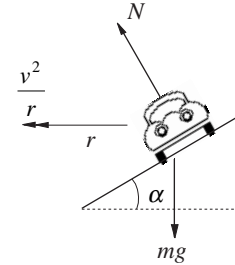
Between these two extremes,  $F$  can be zero. This is important because of the reduced wear on the car tyres and the greater comfort for passengers and driver.

Consider the critical speed when friction is zero.

Taking the components of the forces acting on the car in two, mutually perpendicular directions, vertically and radially, and applying Newton's Second Law in each case,

$$\text{vertically upwards,} \quad N \cos \alpha - mg = 0 \quad (1)$$

$$\text{radially inwards,} \quad N \sin \alpha = \frac{mv^2}{r}. \quad (2)$$



Substituting for  $N$  in equation (2) from equation (1) gives

$$\frac{mg \sin \alpha}{\cos \alpha} = \frac{mv^2}{r}$$

$$\Rightarrow \tan \alpha = \frac{v^2}{rg} \quad (3)$$

For a bend of given radius, this equation defines the correct angle of banking so that there is no friction force for a given speed. Alternatively, for a given angle of banking it defines the speed at which the bend should be taken.

### Activity 8 Including the effects of friction

Suppose that the coefficient of sliding friction between the car and the road is  $\mu$ .

Find an expression for the constant speed of a car on a circular banked track of angle  $\alpha$  to the horizontal:

- if the car is on the point of sliding **up** the incline;
- if the car is on the point of sliding **down** the incline.

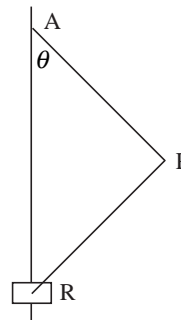
When a railway engine takes the bend on a railway track, what is the effect on the rails of the flanges on the wheels?

### **Exercise 7D**

- A vehicle is approaching a bend which is of radius 50 m. The coefficient of friction between the road and the tyres is 0.5. Find the greatest speed at which the vehicle can safely negotiate the bend if it is horizontal.  
At what angle must the bend be banked if the vehicle is to negotiate the bend without any tendency to slip on the road at a speed of 80 kph?
- A car is negotiating a bend of radius 100 m banked at an angle of  $\tan^{-1}\left(\frac{3}{4}\right)$ . What are the maximum and minimum speeds at which it can do this if the coefficient of friction between the road and the tyres is 0.5?
- A light aeroplane is describing a horizontal circle of radius 200 m at a speed of 150 kph. Calculate the angle at which the plane is banked as it circles, stating the assumptions that you make.
- One lap of a circular cycle track is 400 m, and the track is banked at  $45^\circ$ . At what speed can the track be negotiated without any tendency for skidding or slipping to take place?

## 7.5 Miscellaneous Exercises

- Lucy and Tom ride on a fairground roundabout. Lucy is 2 m and Tom is 1.5 m from the centre of rotation and the roundabout is rotating at 10 revolutions per minute. Find
  - the angular speed of the roundabout in  $\text{rad s}^{-1}$ ,
  - the speeds of Lucy and Tom.
- A marble is made to rotate against the outside rim of a circular tray of radius 0.2 m. If the mass of the marble is 100 g and it is moving at  $2 \text{ ms}^{-1}$ , calculate the horizontal force that the marble exerts on the rim of the tray.
- An athlete throwing the hammer swings the hammer in a horizontal circle of radius 2.0 m. If the hammer is rotating at 1 revolution per second, what is the tension in the wire attached to the hammer, if the mass of the wire is negligible and the mass of the hammer is 7.3 kg?
- A car travels along a horizontal road and can travel without slipping at 40 kilometres per hour around a curve of radius 115 m. Find the coefficient of friction between the tyres and the road surface.
- A train of mass 40 tonnes travels around a banked railway track which forms a circular arc of radius 1250 m. The distance between the rail centres is 1.5 m and the outer rail is 37.5 mm vertically higher than the inner rail. If the train is travelling at 63 kilometres per hour, find the force on the side of the rail. (Use  $g = 9.8 \text{ ms}^{-2}$ ).
- An elastic string, of natural length  $l$  and modulus  $mg$ , has a particle of mass  $m$  attached to one end, the other end being attached to a fixed point O on a smooth horizontal table. The particle moves on the table with constant speed in a circle with centre O and radius  $\frac{3}{2}l$ . Find, in terms of  $g$  and  $l$ , the angular speed of the particle. (AEB)
- A particle moves with constant angular speed around a circle of radius  $a$  and centre O. The only force acting on the particle is directed towards O and is of magnitude  $\frac{k}{a^2}$  per unit mass, where  $k$  is a constant. Find, in terms of  $k$  and  $a$ , the time taken for the particle to make one complete revolution. (AEB)
- One end of light inextensible string of length  $l$  is fixed at a point A and a particle P of mass  $m$  is attached to the other end. The particle moves in a horizontal circle with constant angular speed  $\omega$ . Given that the centre of the circle is vertically below A and that the string remains taut with AP inclined at an angle  $\alpha$  to the downward vertical, find  $\cos \alpha$  in terms of  $l$ ,  $g$  and  $\omega$ . (AEB)
- A particle P is attached to one end of a light inextensible string of length 0.125 m, the other end of the string being attached to a fixed point O. The particle describes with constant speed, and with the string taut, a horizontal circle whose centre is vertically below O. Given that the particle describes exactly two complete revolutions per second find, in terms of  $g$  and  $\pi$ , the cosine of the angle between OP and the vertical. (AEB)
- A particle P of mass  $m$  moves in a horizontal circle, with uniform speed  $v$ , on the smooth inner surface of a fixed thin, hollow hemisphere of base radius  $a$ . The plane of motion of P is a distance  $\frac{a}{4}$  below the horizontal plane containing the rim of the hemisphere. Find, in terms of  $m$ ,  $g$  and  $a$ , as appropriate, the speed  $v$  and the reaction of the surface on P.  
A light inextensible string is now attached to P. The string passes through a small smooth hole at the lowest point of the hemisphere, and has a particle of mass  $m$  hanging at rest at the other end. Given that P now moves in a horizontal circle on the inner surface of the hemisphere with uniform speed  $u$  and that the plane of the motion is now distant  $\frac{a}{2}$  below the horizontal plane of the rim, prove that  $u^2 = 3ga$ . (AEB)
- The figure shows a particle P, of mass  $m$ , attached to the ends of two light inextensible strings, each of length  $l$ . The other end of one string is attached to a fixed point A and the other end of the second string is attached to a ring R of mass  $3m$ . The ring can slide freely on a smooth vertical wire passing through A. The particle P describes a horizontal circle, whose centre is on the wire, with constant angular speed  $\omega$ . The tensions in AP and PR are denoted by  $T_1$  and  $T_2$ , respectively and the angle between AP and the vertical is denoted by  $\theta$ . Given that the ring is at rest find  $T_2$  in terms of  $m$ ,  $g$  and  $\theta$  and hence calculate the ratio  $T_1 / T_2$ .  
Show that  
$$T_1 + T_2 = m\omega^2 l.$$
Hence express  $\cos \theta$  in terms of  $g$ ,  $\omega$  and  $l$  and find the range of possible values of  $\omega$ . (AEB)



12. (In this question take  $g$  to be  $9.8 \text{ ms}^{-2}$ )

A light elastic string of natural length  $0.2 \text{ m}$  has one end attached to a fixed point  $O$  and a particle of mass  $5 \text{ kg}$  is attached to the other end.

When the particle hangs at rest, vertically below  $O$ , the string has length  $0.225 \text{ m}$ . Find the modulus of elasticity of the string.

The particle is made to describe a horizontal circle whose centre is vertically below  $O$ . The string remains taut throughout this motion and is inclined at an angle  $\theta$  to the downward vertical through  $O$ .

- (a) Given that the tension in the string is  $98 \text{ N}$ , find  $\theta$  and the angular speed of the particle.
- (b) Given that the string breaks when the tension in it exceeds  $196 \text{ N}$ , find the greatest angular speed which the particle can have without the string breaking.

(AEB)

13. A rigid pole  $OA$ , of length  $4l$ , has the end  $O$  fixed to a horizontal table with the other end  $A$  vertically above  $O$ . The ends of a light inextensible string, of length  $4l$ , are fixed to  $A$  and a point  $B$  distance  $2l$  below  $A$  on the pole. A small particle of mass  $m$  is fastened to the mid-point of the string and made to rotate, with both parts of the string taut, in a horizontal circle with angular speed  $\omega$ . Find the tension in both parts of the string in terms of  $m$ ,  $l$ ,  $\omega$  and the acceleration due to gravity,  $g$ .

At a given instant both parts of the string are cut. Assuming that there is no air resistance and that  $\omega = \sqrt{(g/3l)}$ , find the time which elapses before the particle strikes the table and show that it does so at a point distant  $3l$  from  $O$ .

(AEB)