# 11 GROWTH AND DECAY

## Objectives

After studying this chapter you should

- understand exponential functions;
- be able to construct growth and decay models;
- recognise graphs of exponential functions;
- understand that the inverse of an exponential function is a logarithmic function;
- be able to use logarithms to solve suitable equations;
- be able to differentiate exponential and logarithmic functions.

## 11.0 Introduction

Amoebae reproduce by dividing after a certain time. Radioactive substances have 'half lives' which are determined by the time it takes the radioactivity to halve. These are examples of systems which are modelled by 'exponential' functions.

The world's human population is growing at about 3% per year. That is after each year the population will be 3% more than it was at the start of the year. In the first activity, you will form a model to describe this population growth, and then use it to find the year when the population will be twice the size it was in 1989.

The population at the end of 1989 was approximately 4.5 billion (4,500,000,000). If the population grew by 3% in 1990, at the end of the year it would be $4.5 \text{ billion} \times 1.03 = 4.635 \text{billion}$.

The population at the end of 1991 can be found by the calculation

$$4.5 \times 1.03 \times 1.03 = 4.5 \times 1.03^2 .$$

By the end of 1992, the population would be $4.5 \times (1.03)^3$.

### Activity 1 Modelling population

For the model described above, calculate the population at the end of each year, starting at 1989, and continuing until the population has doubled from its value at 1989. Plot the values on a graph.

The model used in this activity can be represented by the equation

$$P = P_o a^x \qquad (1)$$

when $P$ is the population at the end of the year number $x$, and $P_o$ is the initial population at year $x = 0$ and $a$ is a constant. So for the world population described above

$$P = 4.5 \times 1.03^x \quad x = 1, 2, 3, \ldots$$

Equations of the form (1) can also be used to describe populations that are declining.

Some species are endangered because they have declining populations, for various reasons like hunting, habitat destruction, new predators or infertility. Many marine species such as whales and some fish give cause for concern. Models are made to help predict future trends in fish stock levels, which take into account many features like fishing techniques and environmental conditions.

In the next activity, you will produce a model of a fish population based on the assumption that it is declining by 15% each year. You then use it to find the number of years before the population becomes so low that it is in danger of being unable to sustain itself.

### Activity 2 Endangered species

Assume that a particular fishery has a population of 100,000 fish and that current fishing methods cause this population to decline by 15% in a year.

(a) Copy and complete the table opposite, and use it to help you form a model of the population $p$ in terms of the years elapsed, $x$, for $x = 1, 2, ..., 10$.

(b) Plot and draw the graph of the fish population for the first 10 years.

(c) If the population falls below say 25 000, the fish become quite widely separated. In these conditions it becomes difficult to find good catches, and the fish themselves breed at a much reduced rate. Therefore, using 25 000 as an 'action level' use your model to find the number of years before which the population becomes dangerously low.

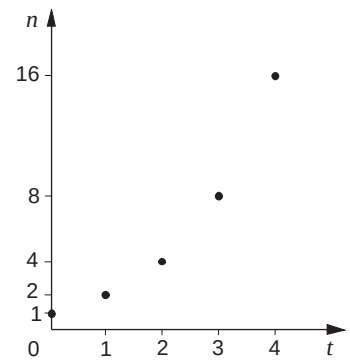
| Years elapsed | Population |
|---|---|
| 0 | 100 000 |
| 1 | 100 000 x 0.85 = 85 000 |
| 2 | … |
| 3 | … |
| 4 | … |
| … | … |

## 11.1 Models of growth and decay

The two activities show how mathematics can be used to model growth and decay of populations. Another example is bacteria, which divides in two every minute. The growth in numbers is illustrated in the following table.

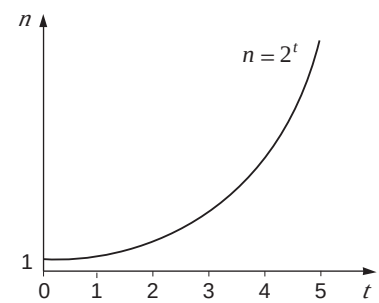
| Minutes past introduction of bacteria | Number of bacteria |
|---|---|
| 0 | $1\ (=2^0)$ |
| 1 | $2\ \left(=2^1\right)$ |
| 2 | $2\times 2=2^2=4$ |
| 3 | $2\times 2\times 2=2^3=8$ |
| 4 | $2\times 2\times 2\times 2=2^4=16$ |
| ... | ... |
| $t$ | $2^t$ |

The table shows a 'pattern ' for the number of bacteria at any time after the bacteria was introduced. From this pattern, it is easy to see that after $t$ minutes the number of bacteria will be $2^t$.




Therefore, $n=2^t$ is a model for the bacteria growth. The graph for this function is shown here, for values of $t$ between 0 and 4.

Drawing a smooth curve through these discrete points gives a continuous model of growth, although in this example, it only makes sense to use whole numbers for $t$. However, many situations which have models like this have continuous not discrete domains, so the curve is typical of this type of function.




Any function of the form $a^x$, where $a$ is a positive constant is called an **exponential** function (as $x$ is the 'exponent' or power of $a$) . Although the example above (and the examples developed in Activities 1 and 2) was only defined for positive values of the exponent ($t\geq 0$), exponential functions are defined for any real value.

For instance, if $f(x)=3^x$, then $f(-1)=3^{-1}=\frac{1}{3}$; using the rules for indices, covered in Chapter 9.

## Activity 3

(a) Use a graphic calculator or computer to help you make sketches of these functions, using the same pair of axes. Use a range of the values between $-3$ and $+3$, and $-30$ to $+30$ on the $y$-axis.

(i) $y=3^x$ (ii) $y=2^x$ (iii) $y=1.5^x$

(iv) $y=1^x$ (v) $y=(0.5)^x$

Do the curves have a common point? What is the relationship between $y = 2^x$ and $y = (0.5)^x$? What happens as $x$ becomes large and positive or large and negative?

(b) Similarly illustrate the graphs of

(i) $y = 3^{-x}$ (ii) $y = 2^{-x}$ (iii) $y = 1.5^{-x}$

and compare them with $y = 3^x, 2^x, 1.5^x$, and describe their behaviour for large $x$, positive or negative.

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Another example of the use of the exponential function is in the modelling of Radon 219, which is an 'isotope' of the gaseous element Radon. It occurs naturally in some types of rock and its seepage from beneath buildings has been identified as a major concern in some parts of the country. Radon 219 is radioactive, with a half life of about 4 seconds. This means that if there are 1000 atoms of Radon 219 in a sample of the gas, 4 seconds later there will be half this number left, 500. 4 more seconds later, and the number of Radon 219 atoms will halve again, to 250, and so on.

This decaying system can be modelled with an exponential function, with a negative exponent.

| Time in seconds after sample is collected (seconds) | Number of atoms left |
|---|---|
| 0 | 1000 |
| 4 | $1000 \times (\frac{1}{2})^1$ |
| 8 | $1000 \times (\frac{1}{2})^2$ |
| 12 | $1000 \times (\frac{1}{2})^3$ |

**Can you write down a formula for $N$, the number of atoms left after time $t$ seconds?**

Since every 4 seconds increases the **power** of the exponent by 1, you can write the model equation as

$$N(t) = 1000 \times (\tfrac{1}{2})^{\frac{1}{4}t}.$$

Note that this can also be written as

$$N(t) = 1000 \times (2^{-1})^{\frac{1}{4}t}$$

$$= 1000 \times 2^{-\frac{1}{4}t},$$

using the properties of indices.

## Activity 4 UK population

The government's statistical service made several predictions about the United Kingdom's population in 1989. One of these was that the population would grow by 1% every ten years. If the population in 1990 was 55 millions, form a model for the UK's future population. Use it to draw a graph of the projected population, and from this estimate the year when the population will equal 60 million.

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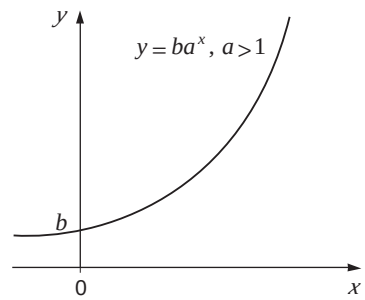
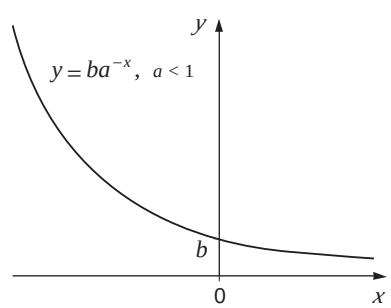



The model used in the activity above is

$$P(t) = P_0 a^{\frac{1}{10}t}$$

when $P_0 = 55(\text{million})$ and $a = 1.01$. So far you have used exponential functions to model various populations and radioactive decay. This last application can be used to help date archaeological objects through Carbon dating.




This section is completed with a summary of the general properties of the exponential functions

$$y = ba^x \text{ and } y = ba^{-x}$$

for $a$>1. As is shown opposite, both curves pass through (0, $b$) and the range of both functions is all real numbers greater than zero. $a$ is the **base** and $x$ the **exponent** of the function.

# 11.2 Carbon dating

Carbon 14 ($^{14}$C) is an isotope of carbon with a half life of 5730 years. It exists in the carbon dioxide in the atmosphere, and all living things absorb some Carbon 14 as they breathe. This remains in an animal or plant, and is constantly added to until the organism dies. After this time, the Carbon 14 decays, reducing to half the amount stored in the body after 5730 years. The amount halves again after another 5730 years, and so on, with no new Carbon 14 absorbed.

In 1946 an American scientist, *Williard Libby*, developed a way of 'dating' archaeological objects by measuring the Carbon 14 radiation present in them. This radioactivity is compared with that found in things living now.

For instance, if bones of recently dead animals produce 10 becquerels per gram of bone carbon (a becquerel is the unit of radioactivity), and an old bone produces only 5 becquerels, the radioactivity has halved since the animal which had the old bone died. As the half life of Carbon 14 is 5730 years, this would mean the animal died in 3740 BC approximately.

## Activity 5 Carbon 14

Complete the table below

| Age (in years) | Radiation (becs) |
|---|---|
| 0 | 10 |
| 5730 | $10 \times \frac{1}{2} = 10 \times 2^{-1}$ |
| 11460 | $10 \times (\frac{1}{2})^2 = 10 \times 2^{-2}$ |
| 17190 | ... |
| 22920 | ... |
| ... | ... |

Use your model to produce a graph, showing radioactivity on the vertical axis and time, in years, on the horizontal. Draw the graph for values of $t$ up to 50,000 years.

From your graph, estimate the ages of bones with these radioactivities

(a) 8.5 becquerels per gram of carbon;

(b) 1.2 becquerels per gram of carbon.

# 11.3 Rate of growth

Suppose a colony of bacteria doubles in number every minute as every member of the colony divides in two. So if there are 2 bacteria at the start of the colony, there will be 4 a minute later (an increase of 2 in one minute), 8 two minutes later (an increase of 4 in one minute) and so on. As the number of bacteria increases, so the **rate** at which that number increases goes up. So the rate of increase of an exponential function is closely related to the value of the function at any point. This suggests that exponential functions and their derivatives are closely linked.

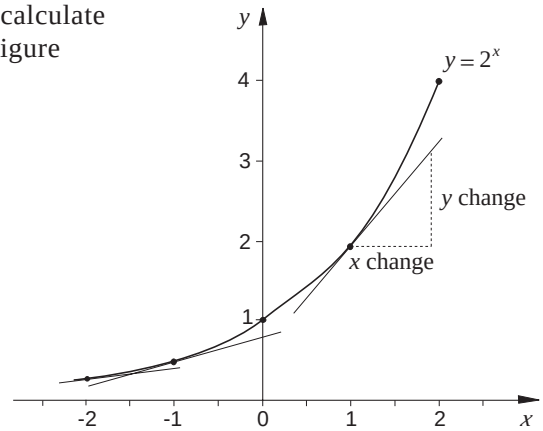
The next activity will explore these links.

## Activity 6

Plot and draw the curves below for values of $x$ between $-2$ and $+2$ and on separate axes.

(a) $y = 2^x$ (b) $y = 3^x$ (c) $y = 2.5^x$ (d) $y = 2.9^x$

Using a ruler to draw tangents to each of your curves, calculate the gradient of each one at five different points. The figure opposite illustrates the method.




Note that

$$\text{gradient} = \frac{\text{change in } y}{\text{change in } x}.$$

In this case all gradients are positive since a positive change in $x$ results in a positive change in $y$.

Plot the five values for the gradient of your graph and sketch in the gradient curve. Comment on how the original graph and the gradient curve seem to be related.

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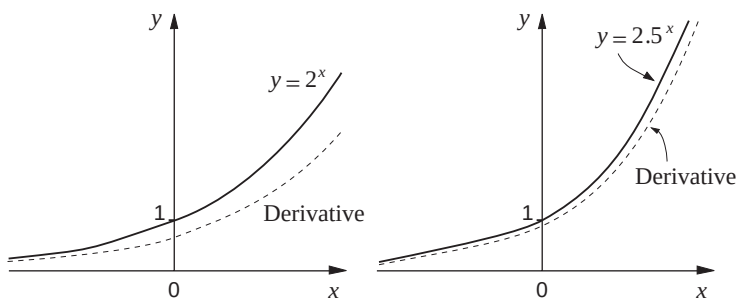
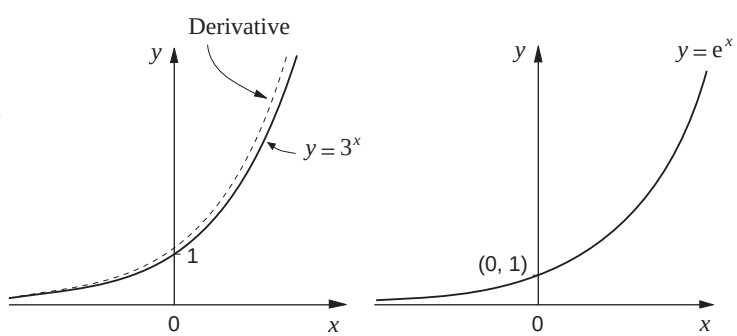
If you have access to a computer or calculator that is capable of showing the derivative of a function, then you can find an exponential function whose derivative exactly fits over its own graph by considering $y = a^x$ with $a$ in the range $2.5 < a < 2.9$.

The derivative of $2^x$ is always less than the value of the function itself. So is the derivative of $2.5^x$, although it is a closer fit to the function than that of $2^x$.




The derivative of $3^x$ has a greater value than the function. This suggests that there is an exponential function, with a base between 2.5 and 3, which has its derivative the same as itself.




Such a function would therefore be its own derivative. The base required for this to happen is denoted by the letter '$e$'.

Unfortunately, its value cannot be given exactly - like $\pi$ and $\sqrt{2}$ it is irrational, and so it can't be expressed exactly as a fraction or decimal. To five decimal places, it is 2.71828.

The function $f(x) = e^x$ is often referred to as the **exponential function**. It is unique in mathematics, in that it is its own derivative. This property makes it extremely important in many branches of the subject.

To summarise

$$\boxed{y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x}$$

## Activity 7

(a) Use a graphic calculator or computer to make sketches of these graphs.

(i) $y = e^x$ (ii) $y = e^{(x+1)}$ (iii) $y = e^{(x-2)}$

(iv) $y = e^x + 1$ (v) $y = e^{-x}$.

(Note that your calculator or computer may use the expression $y = \exp(x)$ for $y = e^x$.)

(b) Compare each of your sketches with the graph $y = e^x$, and state the relationship between each graph and that of $y = e^x$.

(c) Use the fact that the derivative of $e^x$ is $e^x$, to work out the derivatives of each of the other functions.

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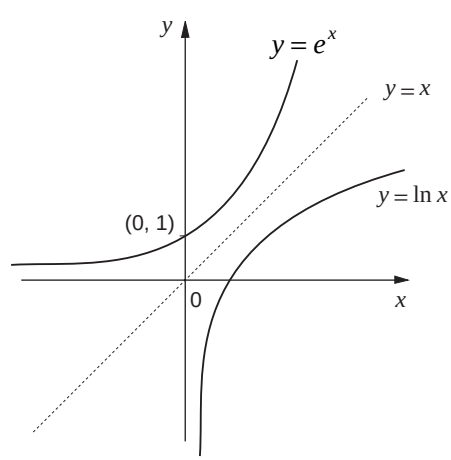
The function $f(x) = e^x$ is a mapping from the set of real numbers, $\mathbb{R}$, to the positive real numbers. Its graph shows that it is a one to one function. This means that $f(x) = e^x$ has an inverse function. The graph of this inverse function is a reflection in the line $y = x$ of the graph of $y = e^x$.




The graph opposite shows $e^x$ and its inverse function, which is usually written as $\ln(x)$. This function is read as 'the natural (or Naperian) logarithm of $x$' or 'the logarithm to base $e$ of $x$'. (*Napier* was a Scottish mathematician of the 16th century who pioneered work connected with this function).

**Why is the domain of $\ln(x)$ only the set of positive real numbers?**

The figure shows that $\ln(x)$ is **not** defined for negative values of $x$ (or zero), as there is no graph to the left of the $y$ axis for $\ln(x)$. So $\ln(-2)$, for instance, does not exist. The range of $\ln(x)$, however, is the full set of real numbers.

## Example

Find $x$ if $e^x = 100$. Give your answer to two d.p.

### Solution

Since $e^x = 100$, and $y = \ln x$ is the inverse function of $e^x$,

$$x = \ln 100$$

Using a calculator to find ln (100), gives $x = 4.61$ to 2 d.p.

To summarise, for $a > 0$,

$$\boxed{e^x = a \Rightarrow x = \ln a}$$

Note that the brackets round '$a$' in ln $a$ have been omitted and will be in future except where it might cause confusion.

## Example

Solve, to 3 s.f. the equation $3e^{2x-1} = 5$.

### Solution

Since $3e^{2x-1} = 5$, then

$$e^{2x-1} = \tfrac{5}{3}$$

Since $e^x$ and $\ln x$ are inverse functions,

$$\begin{aligned} & 2x - 1 = \ln \tfrac{5}{3} \\ \Rightarrow\ & 2x = 1 + \ln \tfrac{5}{3} \\ \Rightarrow\ & x = \tfrac{1}{2}(1 + \ln \tfrac{5}{3}) = 0.755 \text{ to } 3 \text{ s.f.} \end{aligned}$$

# *Exercise 11A*

1. Solve $e^x = 5$ to 2 d.p.
2. Solve $e^x = \frac{1}{2}$ to 2 d.p.
3. Solve $4e^x = 3$ to 3 s.f.
4. Solve $e^{2x} = 1$ to 2 d.p.
5. Solve $3e^{\frac{1}{2}x} = 4$ to 3 s.f.
6. Solve $e^{-x} = 1.5$ to 2 d.p.
7. Solve $4e^{3x-2} = 16$ to 1 d.p.
8. Solve $7e^{3-x} = 2$ to 3 s.f.
9. Solve $e^x \times e^x = 3$ to 2 d.p.
10. Solve $e^{2x} = 4e^x$ to 3 s.f.

# 11.4 Solving exponential equations

Earlier in this chapter, you have produced exponential functions as models, and then used graphs to estimate the solution to a problem. The logarithmic function allows you to calculate rather than estimate these solutions as is shown in the example below.

## Example

A bacteria colony doubles in number every minute, from a starting population of one. The population model is $P = 2^m$, where $P$ is the population and $m$ the number of minutes since the colony was started. Find the time when the population first equals 1000.

### Solution

The problem requires a solution to the equation

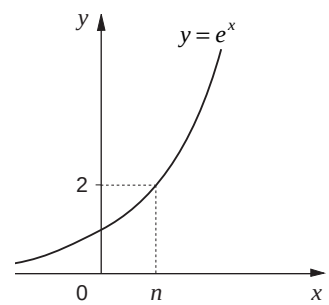
$$P = 1000$$

or

$$2^m = 1000$$

Taking log of each side of the equation

$$\ln 2^m = 1\text{n } 1000 \qquad (1)$$




Now, 2 is a positive real number, so there is some number, call it $n$, such that $e^n = 2$ (see figure opposite). Then $n = \ln(2) \approx 0.693$.

So $2^m = (e^n)^m$ replacing 2 by $e^n$ and since $(e^n)^m = e^{nm}$, using the properties of indices, $2^m$ in (1) above can be replaced by $e^{nm}$, where $n = \ln(2)$.

This gives

$$1\text{n } e^{nm} = 1\text{n } 1000$$

But 1n $x$ is the inverse of $e^x$, so 1n $e^{nm} = nm$. Hence

$$mn = 1\text{n } 1000$$

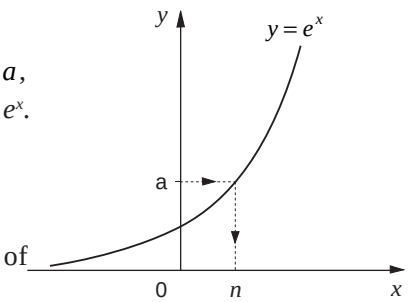
Therefore $m = \dfrac{\ln 1000}{n} = \dfrac{\ln 1000}{\ln 2} = 9.97$ minutes.

Hence $m = 9$ minutes, 58 seconds to the nearest second.

The example illustrates how a general method for solving exponential equations works. This process can be made quicker by using the results developed below.

Consider the function $a^x$, where $a > 0$.

As $a$ is a number greater than zero, there is a real number, $n$, such that $e^n = a$ (see figure opposite). This means that $n = \ln a$, since $\ln x$ is the inverse function for the exponential function $e^x$.




So $a^x = (e^n)^x$ replacing $a$ by $e^n$

That is, $a^x = e^{xn}$ using laws of indices, and taking logarithms of both sides gives the equation

$$\ln a^x = \ln e^{xn} = xn$$

But $n = \ln a$, so

$$\boxed{1\text{n } a^x = x \ln a}$$

This result is a great help in solving a wide variety of exponential equations.

## Example

Solve $3^{2x-1} = 5^x$, giving your answer to 2 d.p.

### Solution

Since $\quad 3^{2x-1} = 5^x$

$$\Rightarrow \quad \ln(3^{2x-1}) = \ln 5^x$$

$$\Rightarrow \quad (2x-1)\ln 3 = x \ln 5$$

$$\Rightarrow \quad 2x \ln 3 - \ln 3 = x \ln 5$$

$$\Rightarrow \quad 2x \ln 3 - x \ln 5 - \ln 3 = 0$$

$$\Rightarrow \quad 2x \ln 3 - x \ln 5 = \ln 3$$

$$\Rightarrow \quad x(2\ln 3 - \ln 5) = \ln 3$$

$$\Rightarrow \quad x = \frac{\ln 3}{2\ln 3 - \ln 5} = 1.87 \text{ to 2 d.p.}$$

## Activity 8

A sample of wood has $^{14}C$ radioactivity of 6 becquerels per gram. New wood has $^{14}C$ radioactivity of 6.68 becquerels per gram of Carbon 14. The half life of $^{14}C$ is 5730 years; form a model based on the work in Section 11.2 for the $^{14}C$ radiation in wood, of the form $R = ba^t$, where $R$ is the radioactivity, $b$ and $a$ are constants, and $t$ is the time in years since the sample was formed.

Use your equation to find to the nearest year when $R = 6$ becquerels per gram of carbon.

## Activity 9

In Activity 4, you used a model for the UK population of the form

$$P = 55 \times 1.01^{\frac{1}{10}t}.$$

$P$ is the population in millions, and $t$ the number of years since 1990. You were asked to estimate the year when the population would first equal 60 million. Solve this problem again by substituting $P = 60$ in the equation, and solving for $t$.

# Exercise 11B

1. Solve $2^x = 5$ to 2 d.p.
2. Solve $3^{\frac{1}{2}x} = 1$ to 2 d.p.
3. Solve $4 \times 2^x = 3$ to 3 s.f.
4. Solve $3^x = 5$ to 2 d.p.
5. Solve $2^{-x} = 6$ to 3 s.f.
6. Solve $3^{2x} = 4$ to 2 d.p.
7. Solve $5^{x-1} = 3$ to 3 s.f.
8. Solve $2^{2x+1} = 4$ to 2 d.p.
9. Solve $5^{x-1} = e^{2x}$ to 1 d.p.
10. Solve $6^{2x+1} = 3^{-x}$ to 2 d.p.

# 11.5 Properties of logarithms

As well as obeying the rule

$$\ln(a^x) = x \ln a,$$

logarithms also obey, for any real numbers $a$, $b$,

$$\ln(ab) = \ln a + \ln b \qquad (1)$$

and

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b \qquad (2)$$

To prove the first result, (1), note that $a$ and $b$ can be written in the form

$$a = e^m, b = e^n$$

for some real numbers $m$ and $n$. Then

$$\begin{aligned}\ln(ab) &= \ln(e^m e^n)\\ &= \ln(e^{m+n})\\ &= m+n\end{aligned}$$

Since $\ln x$ is the inverse function of $e^x$,

$$\ln a = \ln(e^m) = m$$

$$\ln b = \ln(e^n) = n$$

so that

$$\ln(ab) = \ln a + \ln b$$

**How can you deduce equation (2) from (1)?**

You will see how useful these results are in the following applications.

Before the theory of gravitation was developed by *Sir Isaac Newton*, the best laws available to describe planetary motion were those formulated by *Johann Kepler*, a German astronomer. His laws were based on his own meticulous observations, and were used later as a 'benchmark test' for Newton's own theory. This activity investigates Kepler's third law.

## Activity 10 Kepler's third law

This table shows how the average radius of a planet's orbit around the Sun, $R$, is related to the period of that orbit in years, $T$. (The orbits are elliptical, not circular, so an average radius is used here). Only the planets known to Kepler are included.

| Planet | Radius, $R$ (millions of km) | Period, $T$ (years) |
|---|---|---|
| Mercury | 57.9 | 0.24 |
| Venus | 108.2 | 0.62 |
| Earth | 149.6 | 1 |
| Mars | 227.9 | 1.88 |
| Jupiter | 778.3 | 11.86 |
| Saturn | 1427.0 | 29.46 |

You may assume that $T$ and $R$ are linked by a relationship of the form $T = aR^b$ where $a$ and $b$ are constants to be found.

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To fit the model $T = aR^b$ to the data means trying out different values of $a$ and $b$ until you have a good fit with the curve drawn through the data points.

The properties of logarithms will provide us with a better method for finding suitable values of the constants $a$ and $b$.

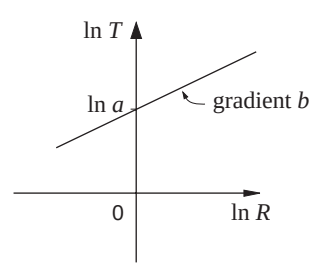
Assume a power law of the form

$$T = aR^b.$$

Taking logs of each side gives

$$\begin{aligned} \ln T &= \ln(a\mathrm{R}^b) \\ &= \ln a + \ln\left(R^b\right) \quad \text{(using equation (1))} \\ &= \ln a + b\ln \mathrm{R} \end{aligned}$$

This equation resembles a straight line equation $y = mx + c$ with $y$ replaced by ln $T$ and $x$ by ln $R$. So a graph of ln $T$ against ln $R$ should give a straight line and the constants $a$ and $b$ can be estimated from the graph. The constant $b$ will be the **gradient** of the line, and ln $a$ will be the **intercept** on the vertical axis.



## Activity 11

For the data in Activity 10, plot a graph of ln $T$ against 1n $R$, and use it to estimate the values of the constants $a$ and $b$.

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The note produced by a musical instrument is directly related to its frequency (the number of times the air is caused to vibrate every second). The higher the frequency, the higher the note. In order to set the frets on a guitar in the correct place, the maker must know how the length of a string affects the frequency of the note it produces.

## Activity 12 Guitar maker's problem

This relationship between length, $l$ (cm), and frequency, $f$ (hz), can be found experimentally. The table shows some data collected by experiment for a particular type of string.

| **Length** $l$ (cm) | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|
| **Frequency** $f$ (hz) | 410 | 330 | 275 | 255 | 225 | 195 |

The relationship is assumed to be of the form $f = al^b$ where $a$ and $b$ are constant.

Use logarithms to 'linearise' the relationship, as described previously. Plot 1n $f$ on a vertical axis and 1n $l$ on the horizontal, and draw a line of best fit. Find the gradient and intercept with the vertical axis of this line, and so determine the values of $a$ and $b$.

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The frequencies produced are also affected by the tension in the string and so, even with frets correctly placed, the guitarist must still 'tune' the instrument by changing the tensions in the strings.

'Middle C' has a frequency of 264 Hz.

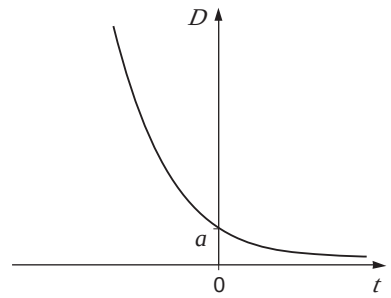
**What length of string gives this frequency?**

The last application in this section is based on the method used by forensic scientists to estimate the time of death of a body.

When a person dies, the body's temperature begins to cool. The temperature of the body at any time after death is governed by Newton's Law of Cooling, which applies to any cooling object:

$$\boxed{D = ae^{-kt}}$$

$D$ is the temperature difference between the cooling object and its surrounding, $a$ and $k$ are constants, and $t$ is the time since the object started to cool. The values of $a$ and $k$ depend on the size, shape and composition of the object and the initial temperature difference.




If $D$ is plotted against the time, $t$, the graph will be similar to the curve shown opposite.

To find out the equation of the curve which applies to a dead body, the values of $a$ and $k$ must be found. This will require two readings of its temperature.

## Example

The police arrive at the scene of a murder at 8 a.m.

On arrival, the temperature of the body and its surroundings are measured at $34°\text{C}$ and $17°\text{C}$ respectively. This was taken to be the moment when the time, $t$, was equal to zero.

At 9 a.m. when $t = 1$, the body temperature was measured as $33°\text{C}$ and the room temperature still as $17°\text{C}$.

Estimate the time of death.

### Solution

The two sets of data are

$$D = 34 - 17 = 17 \text{ at } t = 0$$

$$D = 33 - 17 = 16 \text{ at } t = 1$$

Substituting in the governing equation

$$D = ae^{-kt}$$

gives

$$17 = ae^{-k.0} = ae^{0} = a$$

(since $e^{0} = 1$); and

$$16 = ae^{-k.1} = ae^{-k} = 17e^{-k}.$$

Therefore

$$e^{-k} = \frac{16}{17}$$

and taking 'logs',

$$-k = \ln\left(\frac{16}{17}\right) = -0.0606 \Rightarrow k = 0.0606.$$

Hence

$$D = 17e^{-0.0606t}. \qquad (3)$$

Now normal body temperature is given by $36.9°\text{C}$, so the corresponding value of $D$ is given by

$$D = 36.9 - 17 = 19.9.$$

Substituting this value of $D$ into equation (3) and solving for $t$ will give you the estimated time of death; this gives

$$\begin{aligned}
19.9 &= 17e^{-0.0606t} \\
\Rightarrow \quad e^{-0.0606t} &= \frac{19.9}{17} \\
\Rightarrow \quad -0.0606t &= \ln\left(\frac{19.9}{17}\right) \\
\Rightarrow \quad t &= -\frac{1}{0.0606}\ln\left(\frac{19.9}{17}\right) \\
&= -2.599 \text{ hours} \\
&\approx -(2 \text{ hours } 36 \text{ minutes}).
\end{aligned}$$

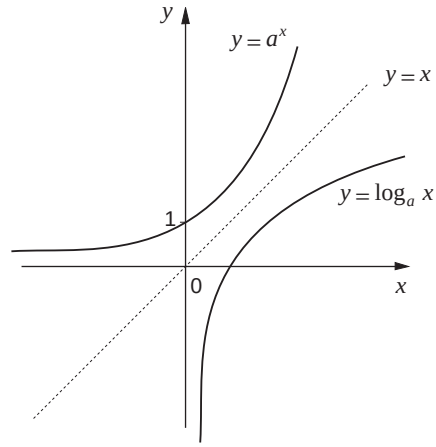
So the estimated time of death is estimated at 5.24 am, or about 5.30 am.

**What important assumptions have been made in this model? Are they reasonable?**

## Activity 13

A body is found at 11.30 pm. The body temperature at midnight is found to be $33°\text{C}$ and at 2.00 am it is $31.5°\text{C}$. Assuming the surroundings are at a constant temperature of $30°\text{C}$, estimate the time of death.

# 11.6 Other bases




Many applications of exponential functions do not use the base $e$. Scientists often use a base of 10 for instance. The graph of $y = a^x$, for $a > 1$, shows that the function of $a^x$ is one to one. This means that it has an inverse function, which is denoted $y = \log_a x$. This is read as "logarithm (or log) to base $a$ of $x$". The figure opposite also shows the graph $y = \log_a x$. The graph also shows that the range of $a^x$ is the positive real numbers, as is the domain of $y = \log_a x$.

Provided logarithms use a suitable base, they obey the same laws developed in earlier sections. That is;

$$\log_a p^n = n \log_a p$$

$$\log_a pq = \log_a p + \log_a q \quad \text{(for any two numbers } p \text{ and } q\text{)}$$

$$\log_a \left(\frac{p}{q}\right) = \log_a p - \log_a q.$$

To summarise;

$$\boxed{\text{if } \; y = a^x, \text{ then } x = \log_a y}$$

## Activity 14

Without using a calculator, answer these questions:

(a) For any base $a$, $a^0 = 1$. Write down $\log_a 1$.

(b) $a^1 = a$. Write down $\log_a(a)$ for any base $a$.

Also write down $\log_a\left(a^2\right)$, $\log_a\left(\sqrt{a}\right)$.

(c) $1000 = 10^3$. Write down $\log_{10}(1000)$.

Similarly, find $\log_{10}(100)$, $\log_{10}\left(\frac{1}{10}\right)$ and $\log_{10}(0.01)$.

(d) What is $\log_2(8)$? (Remember $2^3 = 8$.)

## Exercise 11C

Without using a calculator, answer these questions

1. $\ln e^2$
2. $\log_{10} 10000$
3. $\log_3 27$
4. $\log_2\left(\frac{1}{16}\right)$
5. $\ln\left(\frac{1}{\sqrt{e}}\right)$
6. $\log_{10}\sqrt{10}$
7. $\log_5 125$
8. $\log_{49} 7$

# 11.7 Derivative of ln $x$

You have already seen that if

$$y = e^x \Rightarrow \frac{dy}{dx} = e^x.$$

Now if $y = \ln x$, and $x > 0$

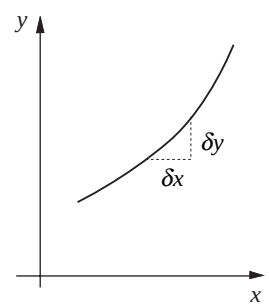
$$x = e^y$$

so that

$$\frac{dx}{dy} = e^y.$$

If $\delta y$ and $\delta x$ are corresponding small changes in $y$ and $x$, then

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right),$$




but $$\frac{dx}{dy} = \lim_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right).$$

Hence $$\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$$

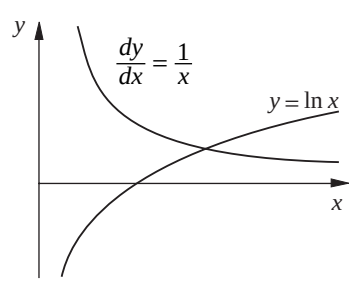
so that, when $y = \ln x$,

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}\ (x > 0).$$

So you have the important result that for $x > 0$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

The function $y = \ln x$ and its derivative are illustrated in the figure opposite. Notice that the graph has only been drawn for values of $x$ greater than zero. This is because $\ln x$ is not defined for negative values of $x$, so it does not have a derivative when $x$ is less than zero.



### Activity 15

Plot the graphs of the following functions using a calculator or a computer. Now that you know the derivative of $\ln(x)$ is $\frac{1}{x}$, try to write down the derivatives of each of these functions, by comparing the curves of each one to $y = \ln(x)$.

(a) $\ln(x+1)$

(b) $\ln(x+2)$

(c) $\ln(x-3)$

(d) $\ln(x)-1$

(e) $-\ln(x)$

If you have a graph plotting package which is capable of displaying the derivatives of each function, you can check your answers.

## 11.8 Miscellaneous Exercises

1. Solve these equations to three significant figures where appropriate:

   (a) $e^x = 4$ (b) $e^{3x} = 0.1$

   (c) $e^{2x-1} = 5$ (d) $3^x = 1$

   (e) $10^x = 5$ (f) $4^x = 5^{2x+1}$

   (g) $3x^2 = 1$ (h) $4 \times 7^{2x} = 6$

   (i) $e \times 2^{x-1} = 5^{1-x}$ (j) $10^x = 1000$

   (k) $5^x = 25$ (l) $2^x = \frac{1}{8}$.

2. A physicist conducts an experiment to discover the half life of an element. The radioactivity at one moment from a sample of the element is measured as 30 becquerels. One hour later the radioactivity is just 28 becquerels. Assuming that the radioactivity is governed by a formula of the form

$$R = a \times 2^{-kt}$$

where $R$ is the radioactivity in becquerels per gram, $t$ the time in hours, and $a$ and $k$ are constants, find the values of $a$ and $k$, and hence determine the half life in hours.